CHAPTER

6

Exponents and Exponential Models

Overview

For studying relationships between sets of numbers, the idea of a mathematical function is central. So far in *Discovering Algebra* students have been studying functions in the guise of linear equations and systems. Before studying the *concept* of function in Chapter 7, students encounter another special case in this chapter: exponential functions. In **Lesson 6.1,** students model exponential growth through recursion with constant multipliers. In **Lesson 6.2,** students discover how to write exponential equations to model the real-life growth being simulated by the recursive routines. Students review the multiplication property of exponents in **Lesson 6.3,** where they are also introduced to the power properties. In **Lesson 6.4,** the multiplication property of exponents is used to introduce and develop scientific notation for large numbers. In **Lesson 6.5,** students review the division property of exponents and then combine exponential expressions involving both multiplication and division. **Lesson 6.6** introduces students to nonpositive exponents as they learn how to use scientific notation for very small numbers. In **Lesson 6.7,** students fit an exponential model to a set of data. **Lesson 6.8** is an activity day in which students do one of two experiments related to exponential decay.

The Mathematics

Exponential Change

Up to this point in *Discovering Algebra,* the phenomena studied have exhibited a constant rate of change. That is, for every unit of change in x, the change in y has been constant. Values can be generated on a calculator by a recursive procedure involving *addition* of a constant at each step.

To model many real-life phenomena, however, values often are best generated on a calculator by recursively *multiplying* by a constant rather than by adding. Instead of consecutive terms having a constant *difference,* they have a constant *ratio.* (A sequence of terms generated by multiplying by a constant is known as a *geometric sequence,* though this terminology isn't used.) The rate of change is not constant but rather is changing in a regular way. The most common examples of these phenomena are financial investments, populations, heating, cooling, and radioactive decay.

A constant rate of change can be represented by an equation of the form $y = a + bx$. The constant b is added to a repeatedly, x times. In contrast, with the multiplicative model, the equation is $y = ab^x$. The constant b multiplies a repeatedly, x times. In this case, the rate of change is not constant, and b isn't the rate of change. In the exponential model, b represents the *constant multiplier* or *growth factor* (even when b is less than 1, in which case the values of y are shrinking rather than growing).

The graph of $y = a + bx$ is a straight line, so the equation represents what is called *linear change.* Because the variable x is an exponent in $y = ab^x$, this kind of change is called *exponential change.* The slope of the graph of $y = ab^x$ is always changing.

For most phenomena exhibiting exponential change, we don't know the constant multiplier directly, but rather we know what portion of y is being added on or removed for each unit change in x. For example, a financial investment earns interest at an annual rate of 5%, or carbon-14 atoms decay at a rate of 35% every 1000 years. In these cases, the constant multiplier b is best written in terms of its relationship to 1. In the case of growth, we write $y = a(1 + r)^x$ and call r the *growth rate.* In the case of decay, we write $y = a(1 - r)^x$ and call r the *decay rate.* The quantities $(1 + r)^x$ and $(1 - r)^x$ are dimensionless numbers being multiplied by the initial value a, which has dimensions.

As a quick way to summarize exponential decay, especially of radioactive substances, scientists often refer to the *half-life* of a particle. The half-life is the amount of change in x that is required to halve the value of y. The fact that the half-life is independent of the initial value of y (it takes just as long to reduce y from 1000 to 500 as from 100 to 50) is an important feature of exponential change.

Going Backward

Just as we often want to solve the linear equation $y = a + bx$ for x in order to predict when growth will reach a certain point, we frequently want to solve the exponential equation $y = ab^x$ for x. That is, we want to find the value of the exponent. There are several ways to do this. To get an approximate value of x, calculator graphs and tables are fine. So is the guess-and-check method—trying various values of x and gradually homing in on one that gives the desired value of y. For a more exact solution, logarithms can be used.

Scientific Notation and Significant Digits

The note on significant digits on page 357 will help you guide students in determining the digits to display in scientific notation or after a decimal point.

Using This Chapter

If your students are already proficient at using the rules of exponents and scientific notation, you can skim or skip Lessons 6.3–6.5. If your class is familiar with nonpositive exponents, you can skip Lesson 6.6. On the other hand, if students are very weak at exponents and you skipped Chapter 0, you might want to do Lessons 0.2 and 0.3 before embarking on this chapter.

Resources

Discovering Algebra Resources

Teaching and Worksheet Masters
Lessons 6.2, 6.3, 6.5, 6.6, 6.7, 6.8

Calculator Notes 0H, 3A, 6A, 6B, 6C, 6D

Fathom Demonstrations
Lessons 6.2, 6.7

CBL 2 Demonstration
Lesson 6.7

Dynamic Algebra Explorations online
Lessons 6.1, 6.2, 6.8

Assessment Resources
Quiz 1 (Lessons 6.1, 6.2)
Quiz 2 (Lessons 6.3–6.6)
Quiz 3 (Lesson 6.7)
Chapter 6 Test
Chapter 6 Constructive Assessment Options

More Practice Your Skills for Chapter 6

Condensed Lessons for Chapter 6

Other Resources

Powers of Ten. Santa Monica, California: Pyramid Film and Video, 1984.

Powers of Ten by Philip and Phylis Morrison.

Functional Melodies by Scott Beall.

Graphic Algebra by Gary Asp et al.

For complete references to these and other resources, see www.keypress.com/DA.

Materials

- graph paper
- paper plates
- protractors
- small counters
- balls
- metersticks
- empty soda cans
- string
- motion sensor, *optional*

Pacing Guide

	day 1	day 2	day 3	day 4	day 5	day 6	day 7	day 8	day 9	day 10
standard	6.1	6.2	6.2	quiz, 6.3	6.4	6.5	6.6	6.6	quiz, 6.7	6.7
enriched	6.1	6.2	6.2, project	quiz, 6.3	6.4	6.5	6.6	6.6	quiz, 6.7	6.7, project
block	6.1, 6.2	6.2, 6.3	6.4, 6.5	6.6	quiz, 6.7	6.8, review	assessment			

	day 11	day 12	day 13	day 14	day 15	day 16	day 17	day 18	day 19	day 20
standard	6.8	review	assessment							
enriched	6.8	review, TAL	assessment							

CHAPTER

6

Exponents and Exponential Models

CHAPTER 6 OBJECTIVES

- Write the exponential form of a sequence generated recursively by a constant multiplier
- Review or learn the multiplication, division, and power properties of exponents
- Move between scientific notation (by hand and on a calculator) and standard notation for numbers
- Rewrite an expression with exponents as an expression with the opposite of those exponents
- Write exponential equations that model real-world growth and decay data

This "Chinese Horse" is part of a prehistoric cave painting in Lascaux, France. Scientific methods that use equations with exponents have determined that parts of the Lascaux cave paintings are more than 15,000 years old. For archaeologists, dating ancient artifacts helps them understand how civilizations evolved. Drawings and pieces of art help them understand what existed at that time and what was important to the civilization. You will see that exponents are useful in many other real-world settings too.

OBJECTIVES

In this chapter you will

- write recursive routines for nonlinear sequences
- learn an equation for exponential growth or decrease
- use properties of exponents to rewrite expressions
- write numbers in scientific notation
- model real-world data with exponential equations

A standard procedure for determining age is by carbon dating (as discussed in the chapter), but many of the paints used in Lascaux are metal-based and don't contain carbon. The claim that the paintings are 15,000 years old is based on carbon dating of paintings done in charcoal and on other artifacts at the site. Non-carbon-based methods indicate that the caves are about 17,000 years old. Students can find more details about these dating methods using the Internet links for Lesson 6.7 available at www.keymath.com/DA. Students might also want to investigate the deterioration of cultural treasures elsewhere, such as in Italy or Egypt.

Knowledge of ancient art has made many modern artists concerned about the durability of the materials they use. Deterioration due to exposure and the growth of organisms in organic fibers and pigments is usually exponential. Students may be interested in the archival methods and materials for preserving family photographs or books. Will the latest film you saw be around in 15,000 years?

LESSON

6.1

Recursive Routines

Slow buds the pink dawn
like a rose
From out night's gray and
cloudy sheath
Softly and still it grows
and grows
Petal by petal, leaf by leaf

SUSAN COOLIDGE

Have you ever noticed that it doesn't take very long for a cup of steaming hot chocolate to cool to sipping temperature? If so, then you've also noticed that it stays about the same temperature for a long time. Have you ever left food in your locker? It might look fine for several days, then suddenly some mold appears and a few days later it's covered with mold. The same mathematical principle describes both of these situations. Yet these patterns are different from the linear patterns you saw in rising elevators and shortening ropes—you modeled those situations with repeated addition or subtraction. Now you'll investigate a different type of pattern, a pattern seen in a population that increases very rapidly.

Investigation

Bugs, Bugs, Everywhere Bugs

You will need

- graph paper

Imagine that a bug population has invaded your classroom. One day you notice 16 bugs. Every day new bugs hatch, increasing the population by 50% each week. So, in the first week the population increases by 8 bugs.

Step 1 In a table like this one, record the total number of bugs at the end of each week for 4 weeks.

Bug Invasion

Weeks elapsed	Total number of bugs	Increase in number of bugs (rate of change per week)	Ratio of this week's total to last week's total
Start (0)	16		
1	24	8	$\frac{24}{16} = \frac{3}{2} = 1.5$
2	36	12	$\frac{36}{24} = \frac{3}{2} = 1.5$
3	54	18	$\frac{54}{36} = \frac{3}{2} = 1.5$
4	81	27	$\frac{81}{54} = \frac{3}{2} = 1.5$

Step 2 The increase in the number of bugs each week is the population's rate of change per week. Calculate each rate of change and record it in your table. Does the rate of change show a linear pattern? Why or why not?

Step 3 Let x represent the number of weeks elapsed, and let y represent the total number of bugs. Graph the data using (0, 16) for the first point. Connect the points with line segments and describe how the slope changes from point to point.

LESSON

6.1

PLANNING

LESSON OUTLINE

One day:

30 min	Investigation
5 min	Sharing
10 min	Examples
5 min	Closing

MATERIALS

- graph paper
- counters, *optional*
- Calculator Note 3A

TEACHING

In some situations values change through multiplying by a constant rather than through adding a constant.

Guiding the Investigation

One Step

"You have a colony of 16 bugs, and the number is increasing by 50% each week. Use your calculators to recursively generate the colony's population for four weeks and to graph the data points. Then decide on a good recursive routine to fit the data." Be sure at least one group thinks about multiplication and not just addition. While groups share a variety of ideas, elicit the notion of the constant multiplier.

Groups might model the growth using counters. Or you may work out the first stage or two with the class.

See page 724 for answers to Steps 2 and 3.

NCTM STANDARDS

CONTENT		PROCESS	
✓	Number		Problem Solving
✓	Algebra	✓	Reasoning
	Geometry		Communication
	Measurement	✓	Connections
✓	Data/Probability	✓	Representation

LESSON OBJECTIVES

- Begin to investigate geometric sequences using recursive routines
- See examples of growth and decay that can be modeled recursively

Step 2 Remind students that a rate of change has dimensions. **[Ask]** "What are the units for the rate of change?" [bugs per week] **[Alert]** If students think the pattern is linear because the rate of change shows a regular pattern, ask what the constant difference is.

Step 3 Students can also be challenged to graph rate of change (y) versus weeks (x) or to find the rate of change of the rate of change. The rate of change of exponential growth is also exponential.

Step 5 $\frac{3}{2}$ or 1.5; multiply the starting number by $\frac{3}{2}$ five times.

Step 6 {0, 16} sets the starting value (when 0 weeks have elapsed) at 16. {Ans (1) + 1, Ans (2) · 1.5} increases the number of weeks elapsed by 1 and multiplies the population value by 1.5.

Step 7 Students will get decimal values for weeks 5 to 8. **[Ask]** "What does a decimal mean when you are talking about population?" [A fraction of a bug cannot exist.] **[Alert]** If students have trouble entering this recursive routine, they may be entering parentheses for braces or vice versa. Remind them to use the 2nd key to make braces around the list {0, 16} and around the entire recursive expression.

Step 7

{1 24}
{2 36}
{3 54}
{4 81}
{5 121.5}
{6 182.25}
{7 273.375}
{8 410.0625}

SHARING IDEAS

Point out the quotation that opens the lesson. Susan Coolidge (1835–1905), also known as Sarah Chauncey Woolsey, was an author of children's literature. This stanza creates an image of growth, a central topic of this chapter.

Ask a student or group to present the table.

Step 4 The constant ratio of $\frac{3}{2}$ indicates that the bug population is multiplied by 1.5 each week. Recursive routines with repeated addition create linear patterns. But here the rate of change between two successive bug populations changes every week. The new values result from multiplication by a constant amount.

Step 4 Calculate the ratio of the number of bugs each week to the number of bugs the previous week, and record it in the table. For example, divide the population after 1 week has elapsed by the population when 0 weeks have elapsed. Repeat this process to complete your table. How do these ratios compare? Explain what the ratios tell you about the bug population growth.

Step 5 What is the **constant multiplier** for the bug population? How can you use this number to calculate the population when 5 weeks have elapsed?

Step 6 Model the population growth by writing a recursive routine that shows the growing number of bugs. [▶ See **Calculator Note 3A** to review recursive routines. ◀] Describe what each part of this calculator command does.

Step 7 By pressing ENTER a few times, check that your recursive routine gives the sequence of values in your table (in the column "Total number of bugs"). Use the routine to find the bug population at the end of weeks 5 to 8.

Step 8 What is the bug population after 20 weeks have elapsed? After 30 weeks have elapsed? What happens in the long run?

Step 8 After 20 wk: 53,204 bugs; after 30 wk: 3,068,017 bugs; possible answer: Natural factors—decreasing resources, death, migration—would cause the population to level off.

In the investigation you found that repeated multiplication is the key to growth of the bug population. Populations of people, animals, and even bacteria show similar growth patterns. Many decreasing patterns, like cooling liquids and decay of substances, can also be described with repeated multiplication.

EXAMPLE A

Maria has saved $10,000 and wants to invest it for her daughter's college tuition. She is considering two options. Plan A guarantees a payment, or return, of $550 each year. Plan B grows by 5% each year. With each plan, what would Maria's new balance be after 5 years? After 10 years?

▶ Solution

With plan A, Maria's investment would grow by $550 each year.

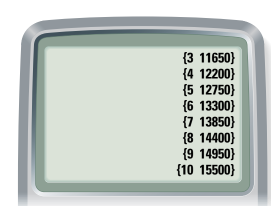
Year	Current balance	+	Return	=	New balance
1	10,000	+	550	=	10,550
2	10,550	+	550	=	11,100
3	11,100	+	550	=	11,650

A recursive routine to do this on your calculator is

{0, 10000} ENTER

{Ans(1)+1, Ans(2)+550} ENTER

ENTER, ENTER, . . .



After 5 years the new balance is $12,750. After 10 years it is $15,500.

[Ask] "How does the population growth of the bugs differ from linear growth? Does it make sense that the more bugs there are, the more will be added each week?" [Yes, there are more bugs reproducing.]

See whether the class can think of other situations in which growth takes place in a "the more there are, the more you get" way. This is a good opportunity to ask students to explain their reasoning without acknowledging the correctness of their answers right away. If someone mentions financial investments, you can go right into Example A.

Assessing Progress

The investigation allows you to assess students' understanding of 50% as $\frac{1}{2}$ and their ability to work with decimals, follow directions systematically, calculate ratios, and enter a recursive sequence.

With plan B, money earns *interest* each year. The amount of interest is 5% of the current balance. To find the new balance at the end of the first year, add the interest to the current balance. Notice that there is a factor of 10,000 in both the current balance and the interest. You can apply the distributive property to write the expression for the new balance in **factored form.**

Year	Current balance	+	Interest (balance × interest rate)	=	New balance (factored form)
1	10,000	+	10,000 × 0.05	=	10,000(1 + 0.05), or 10,500
2	10,500	+	10,500 × 0.05	=	10,500(1 + 0.05), or 11,025
3	11,025	+	11,025 × 0.05	=	11,025(1 + 0.05), or about 11,576

In the first year the balance grows by \$500, to \$10,500. To find the new balance for the next year, you need to add 5% of \$10,500 to the current \$10,500 balance in the account.

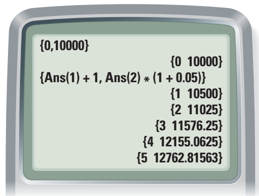
Each year, the balance grows by 5%. To find each new balance, you use the constant multiplier 1 + 0.05, or 1.05.

You can generate the sequence of balances from year to year on your calculator using this recursive routine:

{0, 10000} ENTER

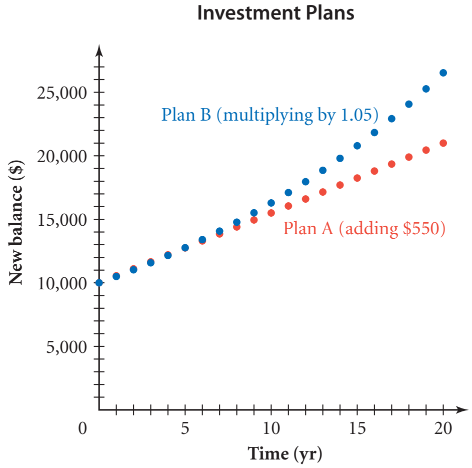
{Ans(1)+1, Ans(2) · (1+0.05)} ENTER

ENTER, ENTER, . . .



The calculator screen shows the sequence of new balances after the first 5 years. Notice that the balance grows by a larger amount each year. That's because each year you're finding a percent of a larger current balance than in the previous year. After 5 years the new balance is about \$12,763. After 10 years it is about \$16,289.

A graph illustrates how the investment plans compare. Given enough time, the balance from plan B, which is growing by a constant percent, will always outgrow the balance from plan A, which has only a constant amount added to it. After 20 years you see an even more significant difference: \$26,533 compared to \$21,000.



▶ EXAMPLE A

This example reinforces the concepts introduced in the investigation and shows how to write exponential growth in factored form. It also compares the results over time for linear and exponential growth. Be sure students understand that an exponential model represents much faster growth than a linear model.

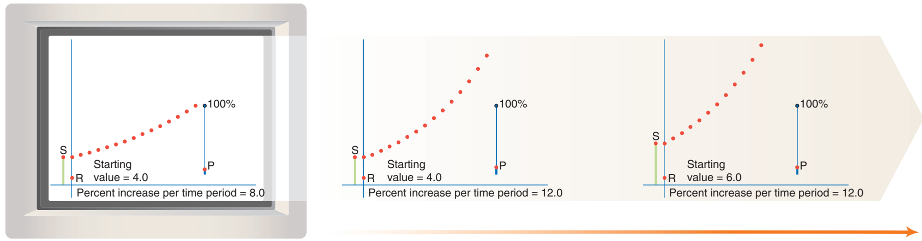
The situation described is not very realistic, because interest is almost always compounded more frequently than once per year. Let students know that the student text is simplifying matters to help with understanding.

The starting balance is often called the *principal* and represented as *P*. Students can think of 1.05 as 105%, or 100% + 5% (principal + interest).

[Alert] Watch for difficulty with the factoring. Merely having students check it by distributing doesn't answer the question "How could I see how to do that?" Point out the occurrence of the same number in each term of the sum and ask what the result would be if you divided each term by that number.

[Ask] "How do banks actually round dollar amounts for balances? How does this compare with what your calculator does?" [A calculator will round up if the decimal represents more than half a cent, but many banks ignore fractions of a cent.]

You might point out that plan A is actually more profitable in the first two years. **[Ask]** "Which plan should Maria choose?" Encourage answers that are based on different assumptions.



keymath.com/DA

[▶The graphs of growth defined by repeated multiplication share certain characteristics. You can use the **Dynamic Algebra Exploration** at www.keymath.com/DA to explore these graphs and to solve some of the exercises in this lesson.◀]

It is helpful to think of a constant multiplier, like 1.05 in Example A, as a sum. The plus sign in $1 + 0.05$ shows that the pattern increases and 0.05 is the percent growth per year, written as a decimal. When a balance or population decreases, say, by 15% during a given time period, you write the constant multiplier as a difference, for example, $1 - 0.15$. The subtraction sign shows that the pattern decreases and 0.15 is the percent decrease per time period, written as a decimal.

Example B uses a proportion and a constant multiplier to calculate a marked-down price.

▶ ***EXAMPLE B***

Encourage students to answer the final question posed in the solution by using a constant multiplier: $34.99(1 - 0.70) \approx 10.50$, so marking down by 70% once gives a different result from two 35% markdowns. Elicit the idea that the second markdown is taking 35% of a smaller amount. Exercise 12 revisits this idea.

EXAMPLE B

Birdbaths at the Feathered Friends store are marked down 35%. What is the cost of a birdbath that was originally priced \$34.99? What is the cost if the birdbath is marked down 35% a second time?

▶ Solution

If an item is marked down 35%, then it must retain 100 − 35 percent of its original price. That is, it will cost 65% of the original price. In Chapter 2, you learned how to set up a proportion using 65% and the ratio of cost, C, to original price.

Cost — Part of the original price retained in sale price

$$\frac{C}{34.99} = \frac{100 - 35}{100} \qquad \text{Write a proportion.}$$

The original price — The whole original price

$$C = \frac{100 - 35}{100} \cdot 34.99 \qquad \text{Multiply by 34.99 to undo the division.}$$

$$C \approx 22.74 \qquad \text{Multiply and divide, and round to the nearest hundredth.}$$

So the cost after the 35% markdown is \$22.74.

You can set up a proportion again to find the cost after the second markdown. Or, you can solve this problem using a constant multiplier. The cost after one 35% markdown is calculated like this:

$$34.99(1 - 0.35) \approx 22.74$$

Using a constant multiplier makes it easy to calculate the cost after the second markdown.

$22.74(1 - 0.35) \approx 14.78$

So the cost after two successive markdowns is $14.78. Is this different than if the birdbath had been marked down 70% one time?

Constant multipliers can be positive or negative. These two sequences have the same starting value, but one has a multiplier of 2 and the other has a multiplier of −2.

3, 6, 12, 24, 48, . . .
3, −6, 12, −24, 48, . . .

How does the negative multiplier affect the sequence? **The negative multiplier changes the sign on every other term. Every other term is the result of multiplying two negatives, and the other terms result from multiplying a negative and a positive.**

EXERCISES

You will need your graphing calculator for Exercises **7** and **9.**

Practice Your Skills

1. Give the starting value and constant multiplier for each sequence. Then find the 7th term of the sequence.

a. 16, 20, 25, 31.25, . . . Ⓐ **starting value: 16; multiplier: 1.25; 7th term: 61.035**

b. 27, 18, 12, 8, . . . **starting value: 27; multiplier: $\frac{2}{3}$, or $0.\overline{6}$; 7th term: $2.\overline{370}$, or $\frac{64}{27}$**

2. Use a recursive routine to find the first six terms of a sequence that starts with 100 and has a constant multiplier of −1.6. **Start with 100, then apply the rule Ans · −1.6; first six terms are 100, −160, 256, −409.6, 655.36, −1048.576.**

3. Write each percent change as a ratio comparing the result to the original quantity. For example, a 3% increase is $\frac{103}{100}$. Then write it as a constant multiplier, for example, 1 + 0.03.

a. 8% increase **$\frac{108}{100}$; 1 + 0.08**

b. 11% decrease **$\frac{89}{100}$; 1 − 0.11**

c. 12.5% growth Ⓐ **$\frac{1125}{1000}$, or $\frac{112.5}{100}$; 1 + 0.125**

d. $6\frac{1}{4}$% loss Ⓐ **$\frac{9,375}{10,000}$, or $\frac{93.75}{100}$; 1 − 0.0625**

e. x% increase **$\frac{100+x}{100}$; $1 + \frac{x}{100}$**

f. y% decrease **$\frac{100-y}{100}$; $1 - \frac{y}{100}$**

4. Use the distributive property to rewrite each expression in an equivalent form. For example, you can write 500(1 + 0.05) as 500 + 500(0.05).

a. 75 + 75(0.02)

b. 1000 − 1000(0.18) Ⓐ

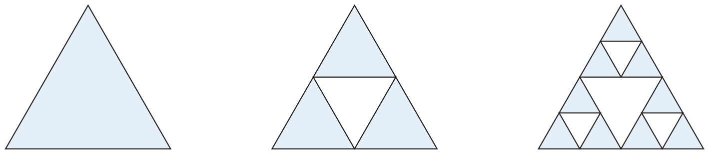
c. $P + Pr$ Ⓐ **$P(1 + r)$**

d. 75(1 − 0.02) **75 − 75 · 0.02**

e. 80(1 − 0.24) **80 − 80 · 0.24**

f. $A(1 - r)$ **$A - A \cdot r$**

5. You may remember from Chapter 0 that the geometric pattern below is the beginning of a fractal called the *Sierpiński triangle.* The Stage 0 triangle below has a total shaded area of 32 square units. Write a recursive routine that generates the sequence of shaded areas in the pattern. Then use your routine to find the shaded area in Stages 2 and 5. Ⓗ



Stage 0
Area = 32 square units

Stage 1
Area = 24 square units

Stage 2

Start with 32, then apply the rule Ans · 0.75; Stage 2 has a shaded area of 18 square units; Stage 5 has a shaded area of 7.59375 square units.

Exercise 4 P is commonly used in applications to mean *principal.* In 4f, A could mean *amount.* In both 4c and 4f, r could mean *rate.* If students try to write answers as single numbers, point out that they haven't factored or distributed.

4a. 75(1 + 0.02), or 75(1.02)

4b. 1000(1 − 0.18), or 1000(0.82)

Exercise 5 Students who have done Chapter 0 can draw Stage 3 (or higher) and calculate the shaded area of this figure. **[Ask]** "In Chapter 0, what mathematical operation did you use to express areas and lengths of fractals?"

Closing the Lesson

When values change through multiplying by a constant rather than through adding a constant, the constant is called a **constant multiplier.** The rate of change is not constant but increases as the amount increases. Examples include growth of populations and the growth of an interest-earning financial investment.

BUILDING UNDERSTANDING

Students practice recursive routines with constant multipliers.

ASSIGNING HOMEWORK

Essential	1–4, 6 or 7–9
Performance assessment	5, 7, 12
Portfolio	9
Journal	11
Group	6, 9
Review	13–16

Helping with the Exercises

Exercise 1 The decay in 1b is represented by having a constant multiplier less than 1.

Exercise 2 In this sequence resulting from a negative multiplier, the terms are neither decreasing nor increasing—they go back and forth between negative and positive.

6a. Start with 20,000, then apply the rule Ans · (1 − 0.04).

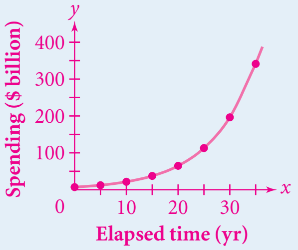
6b. 5th term: 16,986.93; \$16,982.93 is the selling price of the car after four price reductions.

Exercise 6c Students can solve problems like this by using graphs, calculator tables, or guess-and-check.

Exercise 7 [ELL] In the United States, a *billion* is a thousand millions. In other English-speaking countries, a billion is a million millions. "Round to the nearest 0.1 billion" means, in this text, round to the nearest hundred million.

7a. Start with 7.1, then apply the rule Ans · (1 + 0.117).

7c.



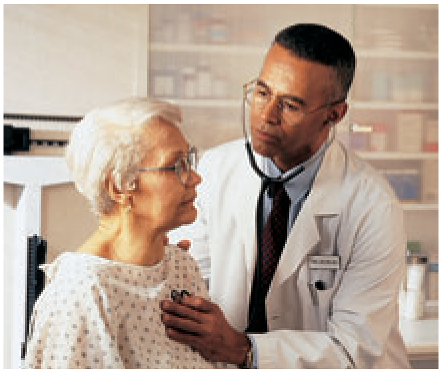
Reason and Apply

6. **APPLICATION** Toward the end of the year, to make room for next year's models, a car dealer may decide to drop prices on this year's models. Imagine a car that has a sticker price of \$20,000. The dealer lowers the price by 4% each week until the car sells.

a. Write a recursive routine to generate the sequence of decreasing prices. @

b. Find the 5th term and explain what your answer means in this situation. @

c. If the dealer paid \$10,000 for the car, how many weeks would pass before the car's sale price would produce no profit for the dealer? 17 wk (the 18th term of the sequence)



7. **APPLICATION** Health care expenditures in the United States exceeded \$1 trillion in the mid-1990s and are expected to exceed \$2 trillion before 2010. Many elderly and disabled persons rely on Medicare benefits to help cover health care costs. According to the Centers for Medicare and Medicaid *2005 Annual Report,* Medicare expenditures were \$7.1 billion in 1970.

a. Assume Medicare spending has increased by 11.7% per year since 1970. Write a recursive routine to generate the sequence of increasing Medicare spending. @

b. Use your recursive routine to find the missing table values. Round to the nearest \$0.1 billion.

Medicare Spending

Year	1970	1975	1980	1985	1990	1995	2000	2005
Elapsed time (yr) x	0	5	10	15	20	25	30	35
Spending (\$ billion) y	7.1	12.3	21.5	37.3	64.9	112.9	196.3	341.3

c. Plot the data points from your table and draw a smooth curve through them.

d. What does the shape of the curve suggest about Medicare spending? Do you think this is a realistic model? Answers will vary. The graph implies a smooth, ever-increasing amount of Medicare spending, which is probably not realistic.

8. **APPLICATION** Ima Shivring took a cup of hot cocoa outdoors where the temperature was 0°F. When she stepped outside, the cocoa was 115°F. The temperature in the cup dropped by 3% each minute.

a. Write a recursive routine to generate the sequence representing the temperature of the cocoa each minute. Start with 115, then apply the rule Ans · (1 − 0.03).

b. How many minutes does it take for the cocoa to cool to less than 80°F? 12 min

9. **APPLICATION** The advertisement for a Super-Duper Bouncing Ball says it rebounds to 85% of the height from which it is dropped.

a. If the ball is dropped from a starting height of 2 m, how high should it rebound on the first bounce? ⓐ 1.7 m

b. Write a recursive routine to generate the sequence of heights for the ball when it is dropped from a height of 2 m. ⓐ Start with 2, then apply the rule Ans • 0.85.

c. How high should the ball rebound on the sixth bounce? approximately 0.75 m

d. If the ball is dropped from a height of 10 ft, how high should it rebound on the tenth bounce? ⓗ approximately 1.97 ft

e. When the ball is dropped from a height of 10 ft, how many times will it bounce before the rebound height is less than 0.5 ft? 19 times

f. A collection of Super-Duper Bouncing Balls was tested. Each ball was dropped from a height of 2 m. The table shows the height of the first rebound for eight different balls. Do you think the advertisement's claim that the ball rebounds to 85% of the original height is fair? Explain your thinking.

Answers will vary. The mean is 1.7 m, or 85% of 2 m, and the median and mode are both 1.68 m, or 84% of 2 m. However, only two of the balls tested met or exceeded 85% of the drop height.

Balls Dropped from 2 m

Ball number	1	2	3	4	5	6	7	8
Height of rebound (m)	1.68	1.67	1.69	1.78	1.64	1.68	1.66	1.8

10. Look back at the six expressions in Exercise 4. Imagine that each expression represents the value of an antique that is increasing or decreasing in value each year. For each expression, identify whether it represents an increasing or decreasing situation, give the starting value, and give the percent increase or decrease per year. ⓗ

11. Grace manages a local charity. A wealthy benefactor has offered two options for making a donation over the next year. One option is to give \$50 now and \$25 each month after that. The second option is to give \$1 now and twice that amount next month; each month afterward the benefactor would give twice the amount given the month before.

a. Determine how much Grace's charity would receive each month under each option. Use a table to show the values over the course of one year. ⓐ

b. Use another table to record the total amount Grace's charity will have received after each month.

c. Let x represent the number of the month (1 to 12), and let y represent the total amount Grace received after each month. On the same coordinate axes, graph the data for both options. How do the graphs compare?

d. Which option should Grace choose? Why?

Exercise 9 You might have students collect data for this exercise using motion sensors attached to calculators. See Lesson 6.8, Experiment 1.

10a. increasing; 75; 2%

10b. decreasing; 1000; 18%

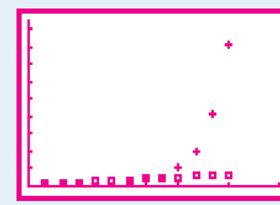
10c. increasing; P; $100r$%

10d. decreasing; 75; 2%

10e. decreasing; 80; 24%

10f. decreasing; A; $100r$%

11c. The graph of the first plan is linear. The graph of the second is not; its slope increases between consecutive points.



[0, 15, 3, 0, 4750, 500]

11d. Possible answer: Grace's charity will receive more total money with option 2. This option will also give the charity more income later in the year when the budget may be tighter.

11a.

	Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec
Option 1	\$50	\$25	\$25	\$25	\$25	\$25	\$25	\$25	\$25	\$25	\$25	\$25
Option 2	\$1	\$2	\$4	\$8	\$16	\$32	\$64	\$128	\$256	\$512	\$1,024	\$2,048

11b.

	Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec
Option 1	\$50	\$75	\$100	\$125	\$150	\$175	\$200	\$225	\$250	\$275	\$300	\$325
Option 2	\$1	\$3	\$7	\$15	\$31	\$63	\$127	\$255	\$511	\$1,023	\$2,047	\$4,095

Exercise 12 This exercise shows that a 3.5% increase followed by a 3.5% decrease does not get you back to where you started. Ask students to keep all the decimal places of their calculation on calculators so they won't think the small difference is due to round-off error. This is not an easy concept for students to understand. It highlights that you must always know 3.5% of what, not just 3.5%. Ask students having trouble to start with 200, increase by 10%, then decrease by 10%: 200 + 20 − 22 is not 200. You might also draw a line segment, add 10% to it, and then point out that 10% of the new segment is longer than the amount that was added.

Exercise 14 This exercise reviews Lesson 4.3, but it asks for a line through points with non-integer coordinates.

16c. $y = 45$ for 600 min or less of use; $y = 45 + 0.55(x - 600)$ for more than 600 min of use.

16d. First plan: \$67.50; second plan: \$45.00 (she pays only the flat rate of \$45.00). She should sign up for the second plan.

16e. First plan: \$172.50; second plan: \$182.50. He should sign up for the first plan.

12. **APPLICATION** Tamara works at a bookstore, where she earns \$7.50 per hour.

a. Her employer is pleased with her work and gives her a 3.5% raise. What is her new hourly rate? \$7.76

b. A few weeks later business drops off dramatically. The employer must reduce wages. He decreases Tamara's latest wage by 3.5%. What is her hourly rate now? \$7.49

c. What is the final result of the two pay changes? Explain. ⓗ Her wage has dropped by \$0.01/h because the increase was calculated as 3.5% of \$7.50, but the decrease was based on \$7.76.

Review

4.3 **13.** Write an equation in point-slope form for a line with slope −1.2 that goes through the point (600, 0). Find the y-intercept. $y = -1.2(x - 600)$; y-intercept: (0, 720)

4.3 **14.** Find the equation of the line that passes through (2.2, 4.7) and (6.8, −3.9). $y \approx 4.7 - 1.87(x - 2.2)$, or $y \approx -3.9 - 1.87(x - 6.8)$

3.4 **15.** Match the recursive routine to the equation.

a. $y = 3x + 7$ i
b. $y = -3x + 7$ iii
c. $y = 7x + 3$ ii
d. $y = -7x + 3$ iv

i. Start with 7, then apply the rule Ans + 3.
ii. Start with 3, then apply the rule Ans + 7.
iii. Start with 7, then apply the rule Ans − 3.
iv. Start with 3, then apply the rule Ans − 7.

5.2 **16.** **APPLICATION** A wireless phone service provider offers two calling plans. The first plan costs \$50 per month and offers 500 minutes free per month; additional minutes cost 35¢ per minute. The second plan costs only \$45 a month and offers 600 minutes free per month; but additional minutes cost more—55¢ per minute.

a. Define variables and write an equation for the first plan if you use it for 500 minutes or less. ⓐ Let x represent minutes of use and y represent cost; $y = 50$.

b. Write an equation for the first plan if you use it for more than 500 minutes. ⓐ $y = 50 + 0.35(x - 500)$

c. Write two equations for the second plan similar to those you wrote in 15a and b. Explain what each equation represents.

d. Sydney generally talks on her phone about 550 minutes per month. How much would each plan cost her? Which plan should she choose? ⓐ

e. Louis averages 850 minutes of phone use per month. How much would each plan cost him? Which plan should he choose?

f. For how many minutes of use will the cost of the plans be the same? How can you decide which of these two wireless plans is better for a new subscriber? ⓐ The plans cost the same for 800 min of use. A new subscriber who will use more than 800 min should choose the first plan. If she will use 800 min or less, then the second plan is better.

LESSON

6.2

Exponential Equations

Recursive routines are useful for seeing how a sequence develops and for generating the first few terms. But, as you learned in Chapter 3, if you're looking for the 50th term, you'll have to do many calculations to find your answer. For most of the sequences in Chapter 3, you found that the graphs of the points formed a linear pattern, so you learned how to write the equation of a line.

Recursive routines with a constant multiplier create a different type of increasing or decreasing pattern. In this lesson you'll discover the connection between these recursive routines and exponents. Then, with a new type of equation, you'll be able to find any term in a sequence based on a constant multiplier without having to find all the terms before it.

This sculpture, *Door to Door* (1995), was created by Filipino artist José Tence Ruiz (b 1956) from wood, cardboard, and other materials. The decreasing size of the boxes suggests an exponential pattern.

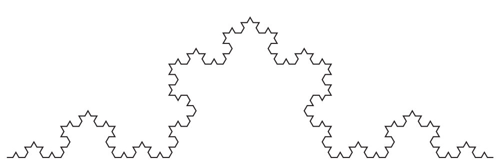
Investigation
Growth of the Koch Curve

You will need

- the worksheet Growth of the Koch Curve

In this investigation you will look for patterns in the growth of a fractal. You may remember the *Koch curve* from Chapter 0. Here you will think about the relationship between the length of the Koch curve and the repeated multiplication you studied in Lesson 6.1. Stage 0 of the Koch curve is already drawn on the worksheet. It is a segment 27 units long.

keymath.com/DA



NCTM STANDARDS

CONTENT		PROCESS	
✓	Number		Problem Solving
✓	Algebra	✓	Reasoning
✓	Geometry	✓	Communication
	Measurement	✓	Connections
	Data/Probability	✓	Representation

LESSON OBJECTIVES

- Explore exponential growth and decay patterns
- Discover the connection between recursive and exponential forms of geometric sequences

LESSON

6.2

PLANNING

LESSON OUTLINE

First day:

40 min Investigation

10 min Sharing

Second day:

30 min Examples

5 min Closing

15 min Exercises

MATERIALS

- Growth of the Koch Curve (W)
- Koch Curve with Sketchpad (W), *optional*
- Calculator Note 6A
- Fathom demonstration Exponential Equations, *optional*

TEACHING

Values changing through multiplication by a constant can be represented using powers of the constant multiplier. The standard form of an exponential, $y = ab^x$, can be found from its linear counterpart $y = a + bx$ by changing the addition to multiplication and the multiplication to exponentiation.

One Step

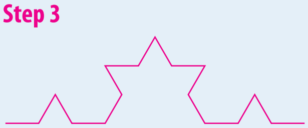
Remind students of the Koch curve from Chapter 0, or introduce it. Ask them to find the constant multiplier for the length when moving from one stage of the curve to the next. Then have them use that multiplier to write expressions for the length of the curve at each of several stages. As you circulate, remind students to use exponents. During Sharing, ask for other examples of exponential growth and have the class make up a problem about a savings account.

Guiding the Investigation

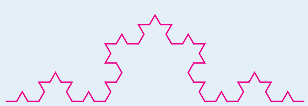
You might want to explore the Koch curve and collect data for Step 2 with The Geometer's Sketchpad and the Koch Curve with Sketchpad worksheet or with the Dynamic Algebra Exploration Koch Curve Growth at www.keymath.com/DA.

Step 1 Students who did not do Chapter 0 may need help in seeing how to generate one stage from another. Have them write a rule. Students may describe something like "Remove the middle third and put a bottomless triangle on it." Alternatively, students can look back at page 14 in Lesson 0.3 to see further stages, or use the How Long Is This Fractal? worksheet from Lesson 0.3. Note that the Stage 0 Koch curve in Chapter 0 does not have length 27.

Step 3



Stage 2



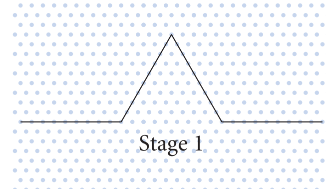
Stage 3

Step 5 Predictions will vary. Exact answers are Stage 4: $27\left(\frac{4}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{3}\right) = 85.\overline{3}$, and Stage 5: $27\left(\frac{4}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{3}\right) = 113.\overline{7}$.

Step 8 Some students may need to be reminded of what exponents are, especially if you skipped Chapter 0. Remember not to say that the exponent gives "the number of times the base is multiplied by itself," because if you multiply the number $\frac{4}{3}$ by itself two times, you'll get $\frac{4}{3} \times \frac{4}{3} \times \frac{4}{3}$, which is $\left(\frac{4}{3}\right)^3$. Rather, say that the exponent gives the number of times the base is a factor in the product.

Step 1 The figure should look like the one shown in the Student Edition, where each segment is 9 units long.

Step 1 Draw the Stage 1 figure below the Stage 0 figure. The first segment is drawn for you on the worksheet. As shown here, the Stage 1 figure has four segments, each $\frac{1}{3}$ the length of the Stage 0 segment.



Stage 1

Step 2 Determine the total length at Stage 1 and record it in a table like this:

Stage number	Total length (units)	Ratio of this stage's length to previous stage's length
0	27	
1	36	$\frac{4}{3} = 1.\overline{3}$
2	48	$\frac{4}{3} = 1.\overline{3}$
3	64	$\frac{4}{3} = 1.\overline{3}$

Step 3 Draw the Stage 2 and Stage 3 figures of the fractal. Again, the first segment for each stage is drawn for you. Record the total length at each stage.

Step 4 Find the ratio of the total length at any stage to the total length at the previous stage. What is the constant multiplier? See table for ratios; $\frac{4}{3}$, or $1.\overline{3}$.

Step 5 Use your constant multiplier from Step 4 to predict the total lengths of this fractal at Stages 4 and 5.

Step 6 twice; $27\left(\frac{4}{3}\right)\left(\frac{4}{3}\right) = 27\left(\frac{4}{3}\right)^2$

Step 6 How many times do you multiply the original length at Stage 0 by the constant multiplier to get the length at Stage 2? Write an expression that calculates the length at Stage 2.

Step 7 three times; $27\left(\frac{4}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{3}\right) = 27\left(\frac{4}{3}\right)^3$

Step 7 How many times do you multiply the length at Stage 0 by the constant multiplier to get the length at Stage 3? Write an expression that calculates the length at Stage 3.

Step 8 If your expressions in Steps 6 and 7 do not use exponents, rewrite them so that they do.

Step 9 Stage 5: $27\left(\frac{4}{3}\right)^5 = 113.\overline{7}$

Step 9 Use an exponent to write an expression that predicts the total length of the Stage 5 figure. Evaluate this expression using your calculator. Is the result the same as you predicted in Step 5?

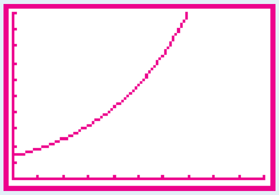
Step 10 Let x represent the stage number, and let y represent the total length. Write an equation to model the total length of this fractal at any stage. Graph your equation and check that the calculator table contains the same values as your table.

Step 11 What does the graph tell you about the growth of the Koch curve? Possible answer: The length of the Koch curve rapidly increases.

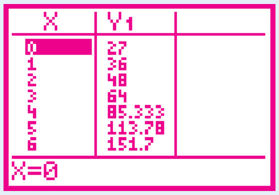
Step 10 **[Ask]** "Are all values of x and y meaningful in this equation?" [Only whole numbers can represent stage numbers and lengths of the first stage of these curves.]

[Alert] Be sure students understand the difference between the fractal itself and the graph that represents its growth.

Step 10 $y = 27\left(\frac{4}{3}\right)^x$



[0, 10, 1, 0, 200, 20]



A recursive routine that uses a constant multiplier represents a pattern that increases or decreases by a constant ratio or a constant percent. Because exponents are another way of writing repeated multiplication, you can use exponents to model these patterns. In the investigation you discovered how to calculate the length of the Koch curve at any stage by using this equation:

Stage number

$$y = 27\left(\frac{4}{3}\right)^x$$

Constant multiplier

Total length Starting length

Equations like this are called **exponential equations** because a variable, in this case x, appears in the exponent. The standard form of an exponential equation is $y = a \cdot b^x$.

When you write out a repeated multiplication expression to show each factor, it is written in **expanded form.** When you show a repeated multiplication expression with an exponent, it is in **exponential form** and the factor being multiplied is called the **base.**

Expanded form **Exponential form**

$$27\underbrace{\left(\frac{4}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{3}\right)}_{} = 27\left(\frac{4}{3}\right)^3$$

The exponent means there are three such factors of $\frac{4}{3}$.

There are three factors of $\frac{4}{3}$.

The base means you are multiplying factors of $\frac{4}{3}$.

EXAMPLE A

Write each expression in exponential form.

a. (5)(5)(5)(5)(5)(5)

b. 3(3)(2)(2)(2)(2)(2)(2)(2)(2)(2)

c. the current balance of a savings account that was opened 7 years ago with $200 earning 2.5% interest per year

▶ Solution

The exponent tells how many times each base is a factor.

a. 5^6

b. There are two factors of 3 and nine factors of 2, so you write

$$3^2 \cdot 2^9$$

You can't combine 3^2 and 2^9 any further because they have different bases.

c. There will be seven factors of $(1 + 0.025)$ multiplied by the starting value of $200, so you write

$$200(1 + 0.025)^7$$

Step 11 Have students consider the slope of the line between consecutive points. They should find that the rate of change is always increasing.

SHARING IDEAS

Select students to share their ideas about Steps 10 and 11.

Assessing Progress
Watch for students' familiarity with fractions and exponents.

▶ EXAMPLE A

This example reviews the meaning of exponents as repeated multiplication.

When the exponent must be a whole number, the variable n is sometimes used instead of x.

▸ *EXAMPLE B*

You can replace this example with the Fathom demonstration Exponential Equations. This example is useful for students who didn't come up with an equation to graph during the investigation. It is also good for refining students' skill at "chunking," seeing a collection of symbols as a single entity. Some students have difficulty seeing $(1 + 0.05)$ as a single quantity that's being multiplied by itself repeatedly. Temporarily rewriting the number as 1.05 might help, but the form $(1 + 0.05)$ emphasizes the fact that the multiplier is more than 1 and the account's value is increasing by 5%.

Students may ask why the constant multiplier is $(1 + 0.05)$ rather than just 0.05. Suggest that they look at $200(0.05)^x$ for several values of x. Then ask them to explain why the 1 is part of the expression.

All dollar values are rounded to the nearest cent. Students' calculators may show different numbers of decimal places, depending on how the mode is set.

Help students distinguish between the growth rate r and the rate of change. The number r is constant and has no units. The rate of change is the amount of increase in y for each unit of increase in x, and it has units.

Closing the Lesson

When values are increasing through multiplication by a constant that's greater than 1, we say that we have **exponential growth** and represent the values by the equation $y = A(1 + r)^x$. You might want to make a bulletin board display of the exponential growth equation.

EXAMPLE B

Seth deposits \$200 in a savings account. The account pays 5% annual interest. Assuming that he makes no more deposits and no withdrawals, calculate his new balance after 10 years.

▸ Solution

The interest represents a 5% rate of growth per year, so the constant multiplier is $(1 + 0.05)$. Now find an equation that you can use to find the new balance after any number of years by considering these yearly calculations and results:

	Expanded form	Exponential form	New balance
Starting balance:	\$200		= \$200.00
After 1 year:	\$200(1 + 0.05)	$= \$200(1 + 0.05)^1$	= \$210.00
After 2 years:	\$200(1 + 0.05)(1 + 0.05)	$= \$200(1 + 0.05)^2$	= \$220.50
After 3 years:	\$200(1 + 0.05)(1 + 0.05)(1 + 0.05)	$= \$200(1 + 0.05)^3$	= \$231.53
After x years:	\$200(1 + 0.05)(1 + 0.05) . . . (1 + 0.05)	$= \$200(1 + 0.05)^x$	

You can now use the equation $y = 200(1 + 0.05)^x$, where x represents time in years and y represents the balance in dollars, to find the balance after 10 years.

$y = 200(1 + 0.5)^x$	Original equation.
$y = 200(1 + 0.05)^{10}$	Substitute 10 for x.
$y \approx 325.78$	Use your calculator to evaluate the exponential expression.

The balance after 10 years will be \$325.78.

Amounts that increase by a constant percent, like the savings account in the example, have **exponential growth.**

Exponential Growth

Any constant percent growth can be modeled by the exponential equation

$$y = A(1 + r)^x$$

where A is the starting value, r is the rate of growth written as a positive decimal or fraction, x is the number of time periods elapsed, and y is the final value.

You can model amounts that decrease by a constant percent with a similar equation. What would need to change in the exponential equation to show a constant percent decrease?

EXERCISES

You will need your graphing calculator for Exercises **5, 7, 9, 11,** and **13.**

Practice Your Skills

1. Rewrite each expression with exponents.

a. (7)(7)(7)(7)(7)(7)(7)(7) 7^8

b. (3)(3)(3)(3)(5)(5)(5)(5)(5) $3^4 \cdot 5^5$

c. $(1 + 0.12)(1 + 0.12)(1 + 0.12)(1 + 0.12)$ @ $(1 + 0.12)^4$

2. A bacteria culture grows at a rate of 20% each day. There are 450 bacteria today. How many will there be

a. Tomorrow? @ $450(1 + 0.2) = 540$ bacteria

b. One week from now?
$450(1 + 0.2)^7 \approx 1612$ bacteria

A technician puts bacteria in several petri dishes of agar. Agar is a gelatin-like substance made from algae. The agar holds the bacteria in place on the petri dish and provides nutrients for growth of the bacteria.

3. Match each equation with a table of values.

a. $y = 4(2)^x$ ii **b.** $y = 4(0.5)^x$ iii **c.** $y = 2(4)^x$ iv **d.** $y = 2(0.25)^x$ i

i.

x	y
0	2
1	0.5
2	0.12
3	0.03

ii.

x	y
0	4
1	8
2	16
3	32

iii.

x	y
0	4
1	2
2	1
3	0.5

iv.

x	y
0	2
1	8
2	32
3	128

4. Match each recursive routine with the equation that gives the same values.

iv **a.** 1.05 ENTER
Ans · (0.95) ENTER

ii **b.** 1.05 ENTER
Ans + Ans · 0.05 ENTER

i **c.** 0.95 ENTER
Ans · (1+0.05) ENTER

iii **d.** 0.95 ENTER
Ans · (1−0.05) ENTER

i. $y = 0.95(1.05)^x$

ii. $y = 1.05(1 + 0.05)^x$

iii. $y = 0.95(0.95)^x$

iv. $y = 1.05(1 - 0.05)^x$

BUILDING UNDERSTANDING

Students make connections between recursive routines, explicit definitions, calculator lists, and tables of values.

ASSIGNING HOMEWORK

Essential	2–5, 9, 12
Performance assessment	8, 10, 14
Portfolio	9, 12
Journal	6, 7, 13, 14
Group	12, 13
Review	1, 10, 16

Helping with the Exercises

6. The initial deposit is $500. The account earns 4% interest per year; thus the constant multiplier is $(1 + 0.04)$. The variable x represents the number of years since the initial deposit. The variable y represents the balance after x years.

Exercise 7 In this game, students are deriving only the right-hand expression $A(1 + r)^x$. Yet on paper they should include the $y =$.

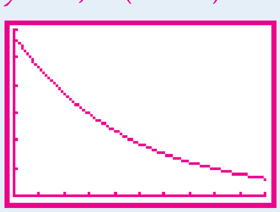
Exercise 9 Make sure students are aware that in actuality, depreciation is not always a constant rate—a car may depreciate 20% one year and 30% another year. Students doing Exercise 13 will see that some goods, such as artwork and antiques, gain in value, or *appreciate.*

9a. $9,200

9b. Start with 11,500, then apply the rule Ans · $(1 - 0.2)$.

9d. $y = 11{,}500(1 - 0.2)^x$

9e.



[0, 10, 1, 0, 12000, 2000]

5. For each table, find the value of the constants a and b such that $y = a \cdot b^x$. (*Hint:* To check your answer, enter your equation into Y1 on your calculator. Then see if a table of values matches the table in the book.)

a.

x	y
0	1.2
1	2.4
2	4.8
3	9.6
4	19.2

$y = 1.2 \cdot 2^x$

b. @

x	y
0	500
2	20
3	4
5	0.16
7	0.0064

$y = 500 \cdot 0.2^x$

c.

x	y
3	8
1	50
5	1.28
2	20
7	0.2048

$y = 125 \cdot 0.4^x$

6. The equation $y = 500(1 + 0.04)^x$ models the amount of money in a savings account that earns annual interest. Explain what each number and variable in this expression means.

7. Run the calculator program INOUTEXP and play the easy-level game five times. Each time you play, write down the input and output values you were given and the exponential equation that models those values. [▶ See **Calculator Note 6A** for instructions on running the program INOUTEXP. ◀] You may wish to team up with another student and use one calculator to run the program while using another calculator to find the constant multiplier. **Students will self-check their work as they run the program.**

Reason and Apply

8. **APPLICATION** A credit card account is essentially a loan. A constant percent interest is added to the balance. Stanley buys $100 worth of groceries with his credit card. The balance then grows by 1.75% interest each month. How much will he owe if he makes no payments in 4 months? Write the expression you used to do this calculation in expanded form and also in exponential form. @
$100(1 + 0.0175)(1 + 0.0175)(1 + 0.0175)(1 + 0.0175) = 100(1 + 0.0175)^4$; about $107.19

9. **APPLICATION** Phil purchases a used truck for $11,500. The value of the truck is expected to decrease by 20% each year. (A decrease in monetary value over time is sometimes called *depreciation.*)

a. Find the truck's value after 1 year.

b. Write a recursive routine that generates the value of the truck after each year.

c. Create a table showing the value of the truck when Phil purchases it and after each of the next 4 years.

d. Write an equation in the form $y = A(1 - r)^x$ to calculate the value, y, of the truck after x years.

e. Graph the equation from 9d, showing the value of the truck up to an age of 10 years.

Many people, like these ranch workers in Montana, rely on a truck for work and leisure.

9c.

Time elapsed (yr)	0	1	2	3	4
Value ($)	11,500	9,200	7,360	5,888	4,710.40

10. Draw a “starting” line segment 2 cm long on a sheet of paper.

a. Draw a segment 3 times as long as the starting segment. How long is this segment? 6 cm

b. Draw a segment 3 times as long as the segment in 10a. How long is this segment? 18 cm

c. Use the starting length and an exponent to write an expression that gives the length in centimeters of the next segment you would draw. @ $2(3)^3$

d. Use the starting length and an exponent to write an expression that gives the length in centimeters of the longest segment you could draw on a 100 m soccer field. $2(3)^7$

11. Run the calculator program INOUTEXP and play the medium- or difficult-level game five times. Each time you play, write down the input and output values you were given and the exponential equation that models those values. [▶ See Calculator Note 6A for instructions on running the program INOUTEXP. ◀] You may wish to team up with another student and use one calculator to run the program while using another calculator to find the constant multiplier.

Students will self-check their work as they run the program.

12. Fold a sheet of paper in half. You should have two layers. Fold it in half again so that there are four layers. Do this as many times as you can. Make a table and record the number of folds and number of layers.

a. As you fold the paper in half each time, what happens to the number of layers?

b. Estimate the number of folds you would have to make before you have about the same number of layers as the number of pages in this textbook.

c. Calculate the answer for 12b. You may use a recursive routine, the graph or table of an equation, or a trial-and-error method.

Origami is the Japanese art of paper folding. To learn more about the history and mathematics of origami, see the links at www.keymath.com/DA .

13. APPLICATION Phil’s friend Shawna buys an antique car for \$5,000. She estimates that it will increase in value (*appreciate*) by 5% each year.

a. Write an equation to calculate the value, y, of Shawna’s car after x years. @ $y = 5000(1 + 0.05)^x$

b. Simultaneously graph the equation in 13a and the equation you found in 9d. Where do the two graphs intersect? What is the meaning of this point of intersection? @

14. Invent a situation that could be modeled by each equation below. Sketch a graph of each equation, and describe similarities and differences between the two models.

$y = 400 + 20x$

$y = 400(1 + 0.05)^x$

15. Consider the recursive routine

{0, 100} ENTER

{Ans(1)+1, Ans(2) · (1−0.035)} ENTER

a. Invent a situation that this routine could model. Answers will vary.

b. Create a problem related to your situation. Carefully describe the meaning of the numbers in your problem. Answers will vary.

c. Use an exponential equation to solve your problem. Answers will vary. The solution should use $y = 100(1 - 0.035)^x$.

12a. The number of layers doubles with each fold.

12b. Estimates will vary.

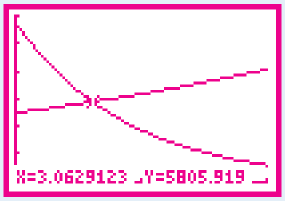
12c. Methods will vary. Eight folds give 256 layers (512 pages), and nine folds give 512 layers (1024 pages).

Exercise 13 This is the first time students may notice that in $y = A(1 + r)^x$ the value of x can be fractional. The graph of the equation is continuous, not discrete.

Students may not know how to convert a decimal number of years to years and months. They can multiply 3.06 years by 12 months per year to get approximately 37 months, or 3 years and 1 month.

If students use trace to find the intersection of the graphs to answer 13b, the width of the pixels on the screen may prevent them from finding the exact position.

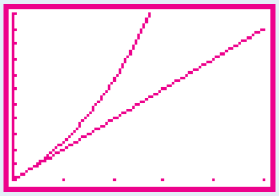
13b.



[0, 10, 1, 0, 12000, 2000]

The intersection point represents the time and the value of both cars when their value will be the same. By tracing the graph shown, students should see that both cars will be worth approximately \$5,800 after a little less than 3 years 1 month.

14. Possible answer: The first equation could model a principal of \$400 to which \$20 is added each time period; the second equation could model a starting bank balance of \$400, with 5% interest added to the balance each time period. Both models have the same starting value, 400. In both models, $y = 420$ when $x = 1$. For x greater than 1, y increases much more quickly in the second model.



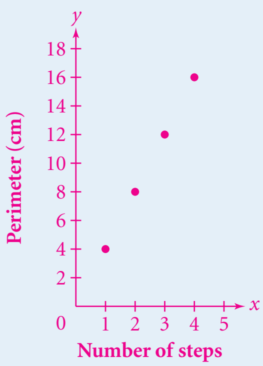
[0, 50, 10, 400, 1500, 100]

Exercise 16 This situation can be modeled with colored tiles or centimeter cubes.

16a.

Number of steps x	1	2	3	4
Perimeter (cm) y	4	8	12	16

16b.

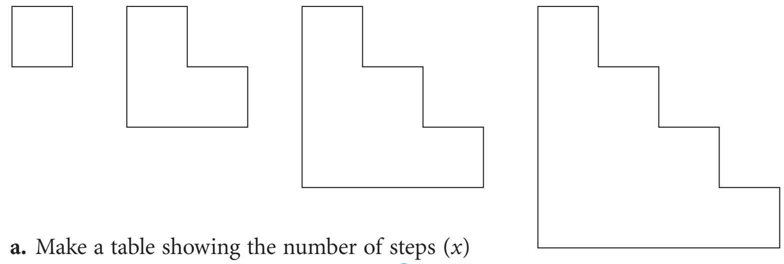


Student Data for Project

Data should be collected from *Kelly Blue Book* or reliable online resources with consideration given to features of the car (should be similar for all years) and mileage (should increase as age increases). Students may tend to notice a linear pattern, depending on the actual rate of depreciation and how far back the data go. If factors such as mileage and features are not closely considered, the data may appear to have no pattern.

Review

3.1 **16.** Look at this "step" pattern. In the first figure, which has one step, each side of the block is 1 cm long.



a. Make a table showing the number of steps (x) and the perimeter (y) of each figure. @

b. On a graph, plot the coordinates your table represents.

c. Write an equation that relates the perimeter of these figures to the number of steps. $y = 4x$

d. Use your equation to predict the perimeter of a figure with 47 steps. 188 cm

e. Is there a figure with a perimeter of 74 cm? If so, how many steps does it have? If not, why not? A perimeter of 74 cm is not possible because it would require 18.5 steps.

project

AUTOMOBILE DEPRECIATION

Cars usually lose value as they get older. Dealers and buyers may rely on books or Internet resources to help them find out how much a used car is worth. But many people don't understand what type of math is used to make these judgments.

Choose a model of automobile that has been manufactured for several years. Research the new-car value now. Then research how much the same model would be worth now if it were manufactured last year, the year before that, and so on. Do your data show a pattern? If so, write an equation that models your data.

Your project should include

- Data for at least 10 consecutive years.
- A scatter plot comparing age and value.
- The rate of change (if your data appear linear) or the rate of depreciation as a percent (if your data look exponential).
- An equation that fits your data.
- A summary of your procedures and findings; include how you collected your data and how well your equation fits the data.

You might want to ask a local auto dealership how it determines a car's value. Does it use the same rate of depreciation for all cars? And how do special features, like a custom stereo, affect the value?

Supporting the project

MOTIVATION

The cost of driving a car includes more than gas and repairs. It also includes the depreciation, the difference between what was paid for the car and what it can be sold for.

OUTCOMES

- The data and scatter plot correctly illustrate the findings.
- The report includes a summary of procedures and findings, including information about how the data were found.
- An equation in the form $y = a + bx$ or $y = A(1 - r)^x$ should be stated and shown to be a good model of the data.
- The report includes the rate of depreciation.
- Appropriate adjustments are made to the equations to achieve a good fit.
- The report includes extended research on the equations actually employed in the used-car industry and how they are applied.

LESSON

6.3

Multiplication and Exponents

Growth for the sake of growth is the ideology of the cancer cell.

EDWARD ABBEY

In Lesson 6.2, you learned that the exponential expression $200(1 + 0.05)^3$ can model a situation with a starting value of 200 and a rate of growth of 5% over three time periods. How would you change the expression to model five time periods, seven time periods, or more? In this lesson you will explore that question and discover how the answer is related to a rule for showing multiplication with exponents.

Every year, the population of the United States increases. This photo shows Grand Central Station in New York City, which is the most populated U.S. city.

Social Science CONNECTION

The U.S. Bureau of the Census only collects population information every 10 years. It uses mathematical models, like exponential equations, to make population predictions between census years.

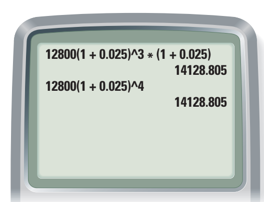
Suppose the population of a town is 12,800 and the town's population grows at a rate of 2.5% each year.

An expression for the population 3 years from now is $12{,}800(1 + 0.025)^3$. To represent one more year, you can write the expression $12{,}800(1 + 0.025)^4$. You can also think about the growth from 3 years to 4 years recursively. Because the rate of growth is constant, multiply the expression for 3 years by one more constant multiplier to get $12{,}800(1 + 0.025)^3 \cdot (1 + 0.025)^1$.

This means that

$$12{,}800(1 + 0.025)^3 \cdot (1 + 0.025)^1 = 12{,}800(1 + 0.025)^4$$

Both methods make sense and both evaluate to the same result.



So you can advance exponential growth one time period either by multiplying the previous amount by the base (the constant multiplier) or by increasing the exponent by one. Every time you increase by one the number of times the base is used as a factor, the exponent increases by one. But what happens when you want to advance the growth by more than one time period? In the next investigation you will discover a shortcut for multiplying exponential expressions.

NCTM STANDARDS

CONTENT		PROCESS	
✓	Number		Problem Solving
✓	Algebra	✓	Reasoning
	Geometry		Communication
	Measurement	✓	Connections
	Data/Probability	✓	Representation

LESSON OBJECTIVES

- Review or learn the multiplication property of exponents
- Review or learn the power properties of exponents

LESSON

6.3

PLANNING

LESSON OUTLINE

One day:

5 min	Introduction
15 min	Investigation
5 min	Sharing
15 min	Examples
5 min	Closing
5 min	Exercises

MATERIALS

- Properties of Exponents (T), *optional*
- Calculator Note 6B

TEACHING

Exponential growth models motivate rules for exponents when multiplying quantities with the same bases and when raising a power to a power.

INTRODUCTION

To estimate a population between actual counts, the growth rate is assumed to be the same between the counts. Theoretically, the graph of a population is a step function, because the population is counted by whole numbers. Because the steps would be so small in the graph, the best model assumes that the population takes on non-integer values and grows continuously. In estimating the population, students should round to the nearest integer.

[Alert] Some students may have forgotten that $(1 + 0.025)^1$ equals $1 + 0.025$. It's sometimes useful to write a number as itself to the first power when trying to see or demonstrate a pattern.

Guiding the Investigation

One Step
Say that the number of ants in a colony has been growing exponentially for 5 wk and has reached a value of $16(1 + 0.5)^5$. Assuming that the growth will continue at the same rate, what will the population be 3 wk from now? As you circulate, ask students to write an expression for that population in at least two ways—$16(1 + 0.5)^5(1 + 0.5)^3$ and $16(1 + 0.5)^8$. **[Ask]** "Can you always add exponents when multiplying?" [Bring out the necessity of having the same base.] Ask students to make up and solve some similar problems with which to challenge each other.

Step 1 Having students go through the "expanded form" will help them make sense of and remember the multiplication property of exponents. Students may have difficulty seeing $1 + 0.05$, the base, as a "chunk."

Step 2 You may want to extend this instruction to include "Write an explanation of this pattern for a friend who is absent today."

Step 3 **[Ask]** "What would happen in something like a^3b^2?" [Exponents can't be added if the bases differ. If the bases are the same, we say they are *like bases*.]

Step 4 The real-world contexts help students develop understanding of the multiplication rule for exponents. The contexts should also help students make the connection with the exponential growth and decay models.

Investigation
Moving Ahead

Step 1c $[(1 + 0.05) \cdot (1 \cdot 0.05)] \cdot [(1 + 0.05) \cdot (1 + 0.05) \cdot (1 + 0.05) \cdot (1 + 0.05)] = (1 + 0.05)^6$

Step 2 Sample answer: You add the original exponents to get the exponent on the final expression.

Step 1 Rewrite each product below in expanded form, and then rewrite it in exponential form with a single base. Use your calculator to check your answers.

a. $3^4 \cdot 3^2$ $(3 \cdot 3 \cdot 3 \cdot 3) \cdot (3 \cdot 3) = 3^6$

b. $x^3 \cdot x^5$ $(x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x \cdot x) = x^8$

c. $(1 + 0.05)^2 \cdot (1 + 0.05)^4$

d. $10^3 \cdot 10^6$ $(10 \cdot 10 \cdot 10) \cdot (10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10) = 10^9$

Step 2 Compare the exponents in each final expression you got in Step 1 to the exponents in the original product. Describe a way to find the exponents in the final expression without using expanded form.

Step 3 Generalize your observations in Step 2 by filling in the blank.

$b^m \cdot b^n = b^{\square}$ $m + n$

Step 4 Apply what you have discovered about multiplying expressions with exponents.

a. The number of ants in a colony after 5 weeks is $16(1 + 0.5)^5$. What does the expression $16(1 + 0.5)^5 \cdot (1 + 0.5)^3$ mean in this situation? Rewrite the expression with a single exponent. the population after 3 more weeks; $16(1 + 0.5)^8$

All ants live in colonies.

b. The depreciating value of a truck after 7 years is $11{,}500(1 - 0.2)^7$. What does the expression $11{,}500(1 - 0.2)^7 \cdot (1 - 0.2)^2$ mean in this situation? Rewrite the expression with a single exponent. the value of the truck after 2 more years; $11{,}500(1 - 0.2)^9$

c. The expression $A(1 + r)^n$ can model n time periods of exponential growth. What does the expression $A(1 + r)^{n+m}$ model? the growth after m more time periods

Step 5 How does looking ahead in time with an exponential model relate to multiplying expressions with exponents?
Sample answer: Multiplying by $(1 + r)^m$ represents looking ahead m time periods.

SHARING IDEAS

Have students share and critique answers to Step 4c.

Draw students' attention to the quotation that opens the lesson. The growth and reproduction of cells is one of many situations that can be modeled by exponential equations.

Ask how good they think predictions will be if made from exponential models of a population. What factors are not taken into account with an exponential model of the town's growth or of the growing cancer cells mentioned in the introduction? In general, exponential models neglect to consider limits on resources (such as the availability of food, jobs, or land) and external interventions (such as immigration or treatments that kill cancer cells).

Assessing Progress
Look for understanding of constant multipliers and of exponents as showing repeated multiplication.

In the investigation you discovered the **multiplication property of exponents.**

Multiplication Property of Exponents

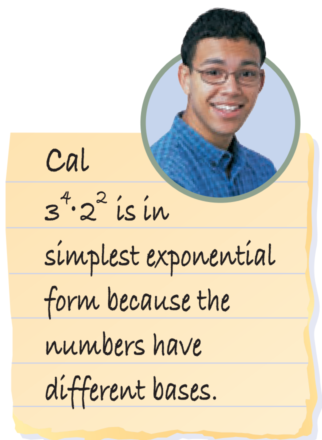
For any nonzero value of b and any integer values of m and n,

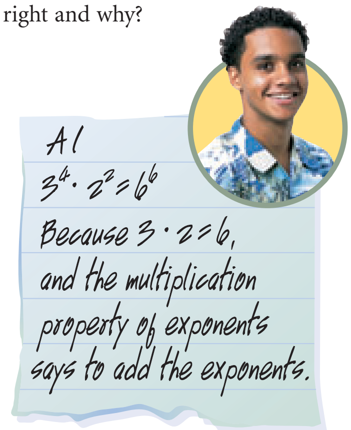
$$b^m \cdot b^n = b^{m+n}$$

This property is very handy for rewriting exponential expressions. However, you can add exponents to multiply numbers only when the bases are the same.

EXAMPLE A

Cal and Al got different answers when asked to write $3^4 \cdot 2^2$ in another exponential form. Who was right and why?





▸ Solution

Rewrite the original expression in expanded form.

$$3 \cdot 3 \cdot 3 \cdot 3 \cdot 2 \cdot 2 = 3^4 \cdot 2^2$$

4 factors of 3   2 factors of 2

The factors are not all the same, so the multiplication property of exponents does not allow you to write this expression with a single exponent. Cal was right. Use your calculator to check that $3^4 \cdot 2^2$ and 6^6 are not equivalent.

EXAMPLE B

Rewrite each expression without parentheses.

a. $(4^5)^2$

b. $(x^3)^4$

c. $(5^m)^n$

d. $(xy)^3$

The multiplication property of exponents does apply if $b = 0$ and m and n are both positive exponents, but not if $b = 0$ and m or n is negative or zero (because this results in a denominator of zero or in 0^0, which is undefined). If b is positive, the property also works for all real values of m and n. Similarly, with the power properties of exponents on page 352, the properties apply if a and b are zero and m and n are positive. If a and b are positive, the properties work for all real values of m and n.

▸ EXAMPLE A

This example stresses that exponents are added only when multiplying powers of the same base. If two factors are raised to the same power, they can be multiplied and the product raised to that same power.

▸ EXAMPLE B

This example derives the power properties of exponents. *Power* is another word for *exponent,* but the terms are used slightly differently. In the expression x^2, the number 2 is the *exponent on* x or the *power on* x. We say that the entire expression is a *power of* x and that the number 27, or 3^3, is a *power of* 3.

If students get confused when doing a problem, suggest rewriting the expression in expanded form and then combining like factors with an exponent.

Another Example

Write $3^4 \cdot 2^2$ with a single exponent. It is not 6^6. But both exponents are even numbers, so you can regroup $3 \cdot 3 \cdot 3 \cdot 3 \cdot 2 \cdot 2$ as $(3 \cdot 3 \cdot 2)(3 \cdot 3 \cdot 2)$, which is 18^2.

▶ Solution

a. Here, a number with an exponent has another exponent. You can say that 4^5 is **raised to the power** of 2. Begin by writing $\left(4^5\right)^2$ as two factors of 4^5.

$$\left(4^5\right)^2 = 4^5 \cdot 4^5 = 4^{5+5} = 4^{10}$$

There is a total of $5 \cdot 2$, or 10, factors of 4.

b. $\left(x^3\right)^4 = x^3 \cdot x^3 \cdot x^3 \cdot x^3 = x^{3+3+3+3} = x^{12}$

There is a total of $3 \cdot 4$, or 12, factors of x.

c. Based on parts a and b, when you raise an exponential expression to a power, you multiply the exponents.

$$\left(5^m\right)^n = 5^{mn}$$

d. Here, a product is raised to a power. Begin by writing $(xy)^3$ as 3 factors of xy.

$$(xy)^3 = xy \cdot xy \cdot xy = x \cdot x \cdot x \cdot y \cdot y \cdot y = x^3y^3$$

Do you remember which property allows you to write $xy \cdot xy \cdot xy$ as $x \cdot x \cdot x \cdot y \cdot y \cdot y$?

This example has illustrated two more properties of exponents.

Power Properties of Exponents

For any nonzero values of a and b and any integer values of m and n,

$$\left(b^m\right)^n = b^{mn}$$
$$(ab)^n = a^nb^n$$

EXERCISES

You will need your graphing calculator for Exercises **1, 8, 9,** and **13.**

▶ Practice Your Skills

1. Use the properties of exponents to rewrite each expression. Use your calculator to check that your expression is equivalent to the original expression. [▶ See **Calculator Note 6B** to learn how to check equivalent expressions. ◀]

a. $(5)(x)(x)(x)(x)$ @ $5x^4$ **b.** $3x^4 \cdot 5x^6$ $15x^{10}$ **c.** $4x^7 \cdot 2x^3$ $8x^{10}$ **d.** $\left(-2x^2\right)\left(x^2 + x^4\right)$ @ $-2x^4 - 2x^6$

2. Write each expression in expanded form. Then rewrite the product in exponential form.

a. $3^5 \cdot 3^8$ **b.** $7^3 \cdot 7^4$ @ **c.** $x^6 \cdot x^2$ **d.** y^8y^5 **e.** $x^2y^4 \cdot xy^3$

3. Rewrite each expression with a single exponent.

a. $\left(3^5\right)^8$ 3^{40} **b.** $\left(7^3\right)^4$ 7^{12} **c.** $\left(x^6\right)^2$ x^{12} **d.** $\left(y^8\right)^5$ y^{40}

4. Use the properties of exponents to rewrite each expression.

a. $(rt)^2$ r^2t^2 **b.** $\left(x^2y\right)^3$ x^6y^3 **c.** $(4x)^5$ $1024x^5$ **d.** $\left(2x^4y^2z^5\right)^3$ $8x^{12}y^6z^{15}$

Closing the Lesson

When **multiplying powers** with like bases, the exponents are added. When **raising a power to a power,** the powers are multiplied. Some students may find it helpful to remember the properties of exponents as "one operation simpler." That is, multiplication means they need to add exponents, while powers means they multiply exponents. The second **power property,** $(ab)^n = a^nb^n$, is sometimes called *distributivity of exponentiation over multiplication.*

Exponentiation and multiplication are analogous in several ways. Exponentiation is repeated multiplication, and multiplication is repeated addition. Exponentiation takes precedence over multiplication in the order of operations, just as multiplication takes precedence over addition. And, just as multiplication distributes over addition, the second power property of exponents says that exponentiation distributes over multiplication. You might show the analogies through examples.

BUILDING UNDERSTANDING

Students practice using the properties of exponents.

ASSIGNING HOMEWORK

Essential	1–7, 13
Performance assessment	6, 8, 12
Portfolio	11, 13
Journal	5, 9, 11
Group	10, 12, 14
Review	15, 16

▶ Helping with the Exercises

Exercise 1 **[Alert]** Some students may propose that $x^2 + x^4 = x^6$. Suggest that they write out the expression in expanded form or look at calculator table values for $Y_1 = x^2 + x^4$ and $Y_2 = x^6$.

Exercises 1 and 8 Students can begin to check their work by substituting numbers for x and y and evaluating.

Exercise 2 Especially if this is the first time students have been asked to multiply exponential expressions with more than one variable, be sure they're not summing exponents for the factors with unlike bases.

2a. $(3 \cdot 3 \cdot 3 \cdot 3 \cdot 3)(3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) = 3^{13}$

2b. $(7 \cdot 7 \cdot 7)(7 \cdot 7 \cdot 7 \cdot 7) = 7^7$

2c. $(x \cdot x \cdot x \cdot x \cdot x \cdot x)(x \cdot x) = x^8$

2d. $(y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y)(y \cdot y \cdot y \cdot y \cdot y) = y^{13}$

2e. $(x \cdot x \cdot y \cdot y \cdot y \cdot y)(x \cdot y \cdot y \cdot y) = (x \cdot x \cdot x) \cdot (y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y) = x^3y^7$

Exercises 2, 4, and 6 If students are not sure how to handle variables without exponents, suggest that they rewrite such variables using the exponent 1.

Reason and Apply

5. An algebra class had this problem on a quiz: "Find the value of $2x^2$ when $x = 3$." Two students reasoned differently.

Student 1 Two times three is six. Six squared is thirty-six.
Student 2 Three squared is nine. Two times nine is eighteen.

Who was correct? Explain why. ⓗ Student 2 was correct; according to the order of operations, squaring should be done before multiplication.

6. Match expressions from this list that are equivalent but written in different exponential forms. There can be multiple matches.

a. $(4x^4)(3x)$ **b.** $(8x^2)(3x^2)$ **c.** $(12x)(4x)$ **d.** $(6x^3)(2x^2)$
e. $12x^6$ **f.** $24x^4$ **g.** $12x^5$ **h.** $48x^2$

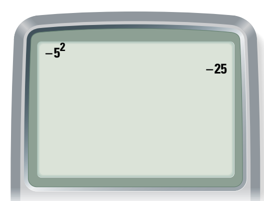
a, d, and g; b and f; c and h; e has no match.

7. Evaluate each expression in Exercise 6 using an x-value of 4.7.
a, d, and g: 27,521.40084; b and f: 11,711.2344; c and h: 1,060.32; e: 129,350.5839

8. Use the properties of exponents to rewrite each expression. Use your calculator to check that your expression is equivalent to the original expression. [▶ See **Calculator Note 6B** to learn how to check equivalent expressions. ◀]

a. $3x^2 \cdot 2x^4$ $6x^6$ **b.** $5x^2y^3 \cdot 4x^4y^5$ $20x^6y^8$ **c.** $2x^2 \cdot 3x^3y^4$ $6x^5y^4$ **d.** $x^3 \cdot 4x^4$ $4x^7$

9. Cal and Al's teacher asked them, "What do you get when you square negative five?" Al said, "Negative five times negative five is positive twenty-five." Cal replied, "My calculator says negative twenty-five. Doesn't my calculator know how to do exponents?" Experiment with your calculator to see if you can find a way for Cal to get the correct answer.

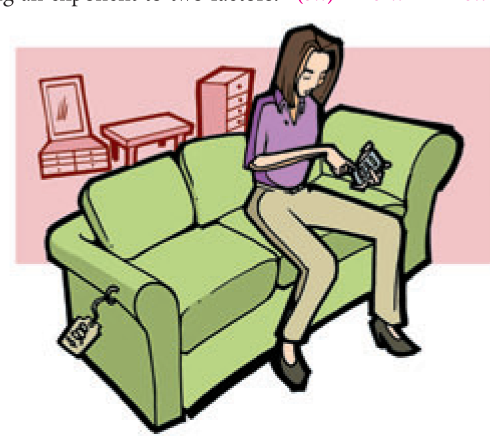


10. Evaluate $2x^2 + 3x + 1$ for each x-value.

a. $x = 3$ ⓐ 28 **b.** $x = 5$ 66 **c.** $x = -2$ 3 **d.** $x = 0$ 1

11. The properties you learned in this section involve adding and multiplying exponents and applying an exponent to more than one factor. Possible answers:

a. Write and solve a problem that requires adding exponents. $x^3 \cdot x^5 = x^8$
b. Write and solve a problem that requires multiplying exponents. $(x^3)^5 = x^{15}$
c. Write and solve a problem that requires applying an exponent to two factors. $(3x)^5 = 3^5x^5 = 243x^5$
d. Write a few sentences describing when to add exponents, when to multiply exponents, and when to apply an exponent to more than one factor.



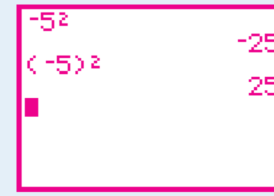
12. **APPLICATION** Lara buys a \$500 sofa at a furniture store. She buys the sofa with a new credit card that charges 1.5% interest per month, with an offer for "no payments for a year."

a. What balance will Lara's credit card bill show after 6 months? Write an exponential expression and evaluate it. ⓐ
$500(1 + 0.015)^6$; \$546.72

Exercises 5 and 8 Students may not recall that the conventional order of operations has exponentiation before multiplication. That is, ab^c means $a(b^c)$ rather than $(ab)^c$. ("Powers have more power.")

Exercise 8 You might ask students to write each expression in *simplest form.* Ask students if there are any common bases remaining in their expression. Make sure that they have multiplied any coefficients to give a final expression with only one coefficient.

9. Enclose the -5 in parentheses.



Exercise 11 Students might also write and solve word problems for 11a–c.

11d. Exponents are added when you multiply two exponential expressions with the same base. Exponents are multiplied when an exponential expression is raised to a power. An exponent is distributed when a product is raised to a power.

b. How much total interest will be added after 6 months? ⓐ $46.72

c. What balance will Lara's credit card bill show after 12 months? Write an exponential expression and evaluate it. $500(1 + 0.015)^{12}$; $597.81

d. How much more interest will be added between 6 and 12 months? $51.09

e. Explain why more interest builds up between 6 and 12 months than between 0 and 6 months. Answers will vary. The increase is greater between 6 and 12 months because the interest each month is a percentage of a greater current balance.

13. Use the distributive property and the properties of exponents to write an equivalent expression without parentheses. Use your calculator to check your answers, as you did in Exercise 1.

a. $x(x^3 + x^4)$ $x^4 + x^5$ **b.** $(-2x^2)(x^2 + x^4)$ $-2x^4 - 2x^6$ **c.** $2.5x^4(6.8x^3 + 3.4x^4)$ ⓗ $17x^7 + 8.5x^8$

14. Write an equivalent expression in the form $a \cdot b^n$. ⓗ

a. $3x \cdot 5x^3$ $15x^4$ **b.** $x \cdot x^5$ x^6 **c.** $2x^3 \cdot 2x^3$ $4x^6$

d. $3.5(x + 0.15)^4 \cdot (x + 0.15)^2$ $3.5(x + 0.15)^6$ **e.** $(2x^3)^3$ $8x^9$ **f.** $[3(x + 0.05)^3]^2$ $9(x + 0.05)^6$

Review

5.5 **15.** Jack Frost started a snow-shoveling business. He spent $47 on a new shovel and gloves. Jack plans to charge $4.50 for every sidewalk he shovels.

a. Write an expression for Jack's profit from shoveling x sidewalks. (*Hint:* Don't forget his expenses.) ⓐ $4.5x - 47$

b. Write and solve an inequality to find how many sidewalks Jack must shovel before he makes enough money to earn back the amount he spent on his equipment.

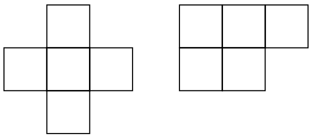
c. How many sidewalks must Jack shovel before he makes enough money to buy a $100 used lawn mower for his summer business? Write and solve an inequality to find out.
$4.5x - 47 > 100$; $x > 32.\overline{6}$; he must shovel 33 sidewalks to pay for his expenses and buy a lawn mower.

5.3 **16.** Solve each system.

a. $\begin{cases} y = 7.3 + 2.5(x - 8) \\ y = 4.4 - 1.5(x - 2.9) \end{cases}$ (5.3625, 0.70625) **b.** $\begin{cases} 2x + 5y = 10 \\ 3x - 3y = 7 \end{cases}$ ⓐ approximately (3.095, 0.762)

IMPROVING YOUR VISUAL THINKING SKILLS

A pentomino is made up of five squares joined along complete sides. The first pentomino can be folded into an open box. The second pentomino can't.



Draw all 12 unique pentominoes, and then identify those that can be folded into open boxes.

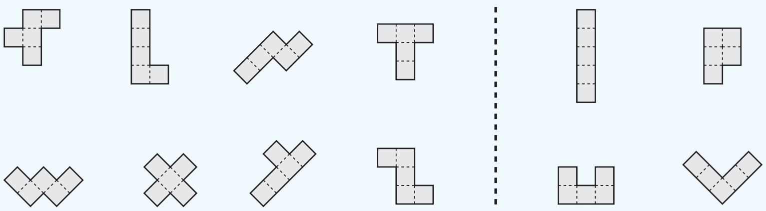
Exercise 15 As an alternative solution for 15b, students can graph the equation $y = 4.5x - 47$, where y represents profit. The x-intercept is Jack's "break-even" point of approximately 11 sidewalks. The point (33, 101.5) represents 33 sidewalks needed to make more than $100 profit.

15b. $4.5x - 47 > 0$; $x > 10.\overline{4}$; he must shovel 11 sidewalks to pay for his equipment.

Exercise 16 This exercise reviews Lesson 5.3, but each solution is a point with non-integer coordinates.

IMPROVING VISUAL THINKING SKILLS

Two pentominos are considered the same if either can be flipped over (reflected) or turned (rotated) to match the other. The 12 distinct, or unique, pentominos are often named after letters of the alphabet they resemble. Of these, the F, L, N, T, W, X, Y, and Z pentominos fold into an open box.



LESSON

6.4

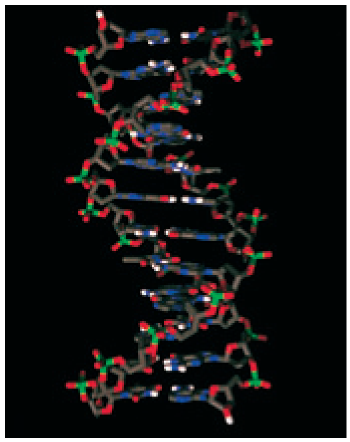
Scientific Notation for Large Numbers

In fact, everything that can be known has number, for it is not possible to conceive of or to know anything that has not.

PHILOLAUS

Did you know that there are approximately 75,000 genes in each human cell and more than 50 trillion cells in the human body? This means that $75{,}000 \cdot 50{,}000{,}000{,}000{,}000$ is a low estimate of the number of genes in your body!

Whether you use paper and pencil, an old-fashioned slide rule, or your calculator, exponents are useful when you work with very large numbers. For example, instead of writing 3,750,000,000,000,000,000 genes, scientists write this number more compactly as 3.75×10^{18}. This compact method of writing numbers is called **scientific notation.** You will learn how to use this notation for large numbers—numbers far from 0 on a number line. The properties of exponents you've learned will help you work with numbers in scientific notation.



This is a computer model of a DNA strand. Many strands of DNA combine to form the genetic information in each cell.

Investigation

A Scientific Quandary

Consider these two lists of numbers:

In scientific notation	Not in scientific notation
3.4×10^5	27×10^4
7.04×10^3	120,000,000
6.023×10^{17}	42.682×10^{29}
8×10^1	4.2×12^6
1.6×10^2	$4^2 \times 10^2$

Step 1 Classify each of these numbers as in scientific notation or not. If a number is not in scientific notation, tell why not.

a. 4.7×10^3 yes **b.** 32×10^5 **c.** $2^4 \times 10^6$
d. 1.107×10^{13} yes **e.** 0.28×10^{11} no; should be 2.8×10^{10}

Step 2 Define what it means for a number to be in scientific notation.

Step 2 Possible answer: The number is factored so that the first factor is a number between 1 and 10 and the second factor is a power of 10.

NCTM STANDARDS

CONTENT		PROCESS	
✓	Number		Problem Solving
✓	Algebra		Reasoning
	Geometry	✓	Communication
✓	Measurement	✓	Connections
	Data/Probability	✓	Representation

LESSON OBJECTIVES

- Write in scientific notation numbers far from zero
- Rewrite in standard notation numbers that are in scientific notation
- Learn how calculators represent scientific notation

LESSON

6.4

PLANNING

LESSON OUTLINE

One day:

5 min	Introduction
20 min	Investigation
5 min	Sharing
5 min	Example
5 min	Closing
10 min	Exercises

MATERIALS

- Calculator Note 6C

TEACHING

This lesson introduces students to scientific notation, both as it's written by hand and as it's represented on a calculator.

One Step

Give the problem from the example and let students struggle for a while with writing all the zeros. Then call them together and ask for ideas about simpler notation. If no one suggests scientific notation, introduce it, have students solve the problem, and let them experiment with scientific notation mode on their calculators to see how to work with it there.

INTRODUCTION

All cells have matching genes, so not all of the 3.75×10^{18} genes in the body are different. The scientific notation assumes that the number of genes given has three significant digits. See page 357 of this teacher's edition for a discussion of significant digits. Page 357 of the student book has a picture of a slide rule.

Step 1b no; should be 3.2×10^6
Step 1c No; the only exponent should be on 10; should be 1.6×10^7.

Guiding the Investigation

Step 1 Scientific notation commonly uses × instead of · to indicate multiplication by a power of 10, because typically numbers rather than letters are being multiplied.

If students aren't familiar with the fact that multiplying by 10 appends zeros or moves the decimal point, you may want to show them a list like this:

$4 \times 10 = 4 \times 10^1 = 40$
$4 \times 10 \times 10 = 4 \times 10^2 = 400$
$4 \times 10 \times 10 \times 10 = 4 \times 10^3 = 4000$

Step 3 *Standard notation* means using numerals and decimal points, with no exponents. You may want to have students also set the number of decimal places from floating to fixed 3. Then 4.7×10^8 will be 4.700E8.

Steps 4 and 5 Many students should do these steps by hand before pulling out their calculators. Calculator displays will vary. Some use *E* (in place of 10) followed by the exponent. Help students translate their display to written scientific notation.

Step 6 **[Alert]** Some students may need more experience with the number of digits in various powers of 10.

SHARING IDEAS

Ask students to share their instructions for Steps 7 and 8. Writing precise instructions is very difficult for many students, so be sympathetic. On the other hand, be sure that weaknesses in the instructions are noted. Work toward the definition that follows Step 8. If appropriate, you might elicit the fact that the condition on a can be written as $1 \leq |a| < 10$.

See page 724 for answers to Steps 4 and 5.

Use your calculator's scientific notation mode to help you figure out how to convert standard notation to scientific notation and vice versa.

Step 3 Set your calculator to scientific notation mode. [▶ See **Calculator Note 6C.** ◀]

Step 4 Enter the number 5000 and press ENTER. Your calculator will display its version, 5×10^3. Use a table to record the standard notation for this number, 5000, and the equivalent scientific notation.

Physicist Suzanne Willis repairs a particle detector at Fermi National Accelerator Lab in Batavia, Illinois. When working with the physics of atomic particles, physicists need scientific notation to write quantities such as 2 trillion electron volts.

Step 5 Repeat Step 4 for these numbers:

a. 250 **b.** −5,530
c. 14,000 **d.** 7,000,000
e. 18 **f.** −470,000

Step 6 In scientific notation, how is the exponent on the 10 related to the number in standard notation? How are the digits before the 10 related to the number in standard notation? If the number in standard notation is negative, how does that show up in scientific notation?

Step 7 Write a set of instructions for converting 415,000,000 from standard notation to scientific notation.

Step 8 Write a set of instructions for converting 6.4×10^5 from scientific notation to standard notation.

Step 6 Answers will vary. The exponent on 10 is the number of digits following the first digit in the original number. The digits before the 10 show the significant digits in the original number. If the number is negative in standard notation, the decimal number will likewise be negative in scientific notation and the minus sign will appear before the digits factor.

Step 7 Possible answer: Write the digits 415 with one digit before the decimal point: 4.15. Determine how many places the decimal point needs to move to have 4.15 become 415,000,000, and make this the exponent on 10. The scientific notation is 4.15×10^8.

Step 8 Possible answer: Move the decimal point in 6.4 five places to the right as represented in 10^5. The standard notation is 640,000.

A number in scientific notation has the form $a \times 10^n$ where $1 \leq a < 10$ or $-10 < a \leq -1$ and n is an integer. In other words, the number is written as a number with one nonzero digit to the left of the decimal point multiplied by a power of 10. The number of digits to the right of the decimal point in a depends on the degree of precision you want to show.

Maria Mitchell (1818–1889) was the first female professional astronomer in the United States.

EXAMPLE

Meredith is doing a report on stars and wants an estimate for the total number of stars in the universe. She reads that astronomers estimate there are at least 125 billion galaxies in the universe. An encyclopedia says that the Milky Way, Earth's galaxy, is estimated to contain more than 100 billion stars. Estimate the total number of stars in the universe. Give your answer in scientific notation.

[Ask] "What real-world quantities could be negative?" [Negative quantities might represent temperatures, location or velocity relative to a fixed point, time before a given time, electrical charges, or acceleration (deceleration).]

Point out the quote introducing the lesson. Philolaus was a philosopher from 475 B.C.E. who lived in what is now southern Italy. Ask students if they agree that everything that can be known for sure can be counted or measured. Scientific notation provides convenient names for numbers far from zero. To motivate Lesson 6.6, you might challenge students to think about how to use scientific notation to describe numbers close to zero.

Assessing Progress

As you observe, you can assess students' skill at seeing patterns, following instructions, applying careful thinking, and working with groups.

▶ Solution

One billion is 1,000,000,000, or 10^9. Write the numbers in the example using powers of 10 and multiply them.

$(125 \times 10^9)(100 \times 10^9)$	125 billion (galaxies) times 100 billion (stars per galaxy).
$125 \times 100 \times 10^9 \times 10^9$	Regroup using the associative and commutative properties of multiplication.
$125 \times 10^2 \times 10^9 \times 10^9$	Express 100 as 10^2.
125×10^{20}	Use the multiplication property of exponents.

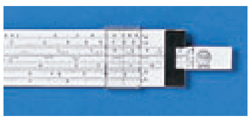
Because 125 is greater than 10, the answer is not yet in scientific notation.

$1.25 \times 10^2 \times 10^{20}$	Convert 125 to scientific notation.
1.25×10^{22}	Use the multiplication property of exponents.

So the universe contains more than 1.25×10^{22} stars.

History CONNECTION

A slide rule is a mechanical device that uses a scale related to exponential notation. Slide rules were widely used for calculating with large numbers until electronic calculators became readily available in the 1970s. To learn more about slide rules, see the links at www.keymath.com/DA .



Notice in this example that you used exponential expressions that were not in scientific notation. Numbers like 125 billion, 100×10^{18}, or 0.03×10^{12} can come up in calculations, and sometimes these numbers make comparisons easier. Scientific notation is one of several ways to write large numbers.

EXERCISES

You will need your graphing calculator for Exercises **8, 10,** and **14.**

Practice Your Skills

1. Write each number in scientific notation.

a. 34,000,000,000 @ 3.4×10^{10} **b.** −2,100,000 -2.1×10^6 **c.** 10,060 1.006×10^4

2. Write each number in standard notation.

a. 7.4×10^4 @ 74,000 **b.** -2.134×10^6 −2,134,000 **c.** 4.01×10^3 4010

3. Use the properties of exponents to rewrite each expression.

a. $3x^5(4x)$ $12x^6$ **b.** $y^8(7y^8)$ @ $7y^{16}$

c. $b^4(2b^2 + b)$ $2b^6 + b^5$ **d.** $2x(5x^3 - 3x)$ $10x^4 - 6x^2$

4. Use the properties of exponents to rewrite each expression.

a. $3x^2 \cdot 4x^3$ $12x^5$ **b.** $(3y^3)^4$ @ $81y^{12}$

c. $2x^3(5x^4)^2$ $50x^{11}$ **d.** $(3m^2n^3)^3$ $27m^6n^9$

5. Owen insists on reading his calculator's display as "three point five to the seventh." Bethany tells him that he should read it as "three point five times ten to the seventh." He says, "They are the same thing. Why say all those extra words?" Write Owen's and Bethany's expressions in expanded form, and evaluate each to show Owen why they are not the same thing.

$3.5 \times 10^7 = 3.5 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 35{,}000{,}000$;
$3.5^7 = 3.5 \cdot 3.5 \cdot 3.5 \cdot 3.5 \cdot 3.5 \cdot 3.5 \cdot 3.5 = 6433.9296875$

Closing the Lesson

Scientific notation provides a way of naming numbers consistently and is especially useful for numbers that are very close to or far from zero. When written, the notation consists of a number between 1 and 10 (possibly 1 but not 10), or between −1 and −10, multiplied by a power of 10. Calculators use other representations.

▶ EXAMPLE

[ELL] Remind students that in the United States the term *billion* means a thousand millions.

Have students check the solution with their calculators. Improving Your Reasoning Skills in Lesson 6.6 will introduce engineering notation, an alternative to scientific notation.

Significant Digits

Significant digits allow you to communicate the degree of rounding in a measurement. For example, if your measuring device showed you only that the length of an object was between 3 and 4 cm, and closer to 3 cm, then you'd report the length as 3 cm. If your device showed you that the length was between 3.0 and 3.1 cm, though, and closer to 3.0, you'd report the length as 3.0 cm.

Using significant digits to report the accuracy of a measurement can be tricky for whole numbers ending in 0. For example, what if your measurement gave you a value between 1200 and 1210, but closer to 1200? You would write 1200. But isn't that also what you'd write if your measurement were between 1200 and 1201, but closer to 1200? Or if your measurement were between 1200 and 1300, but closer to 1200? Just writing 1200 doesn't communicate the number of significant digits.

Here's where scientific notation comes to the rescue. To communicate that 1200 has only two significant digits, you'd write 1.2×10^3. If 1200 has three significant digits, you'd write 1.20×10^3. And to say that it has four significant digits, you'd write 1.200×10^3.

BUILDING UNDERSTANDING

The exercises provide practice with exponents and scientific notation.

ASSIGNING HOMEWORK

Essential	1, 2, 5, 8–10
Performance assessment	6, 11–13
Portfolio	7, 14
Journal	5, 9, 10
Group	14
Review	3, 4, 14, 15

Helping with the Exercises

Exercises 3 and 4 Remind students to rewrite each expression in simplest form.

Exercise 6 Some students may want to use their calculators here. They don't need to use scientific notation mode. They can enter 5.58*10^23, for example, or on some calculators they can enter 5.58 EE23.

Exercise 9 Neither answer is better because the problem doesn't ask for a result in scientific notation.

9a. yes, because they are both equal to 51,800,000,000

9d. Rewrite the digits before the 10 in scientific notation, then use the multiplication property of exponents to add the exponents on the 10's. In this case, $4.325 \times 10^2 \times 10^3 = 4.325 \times 10^5$.

10b. Regroup, multiply the numbers, and multiply the powers of 10 by adding the exponents.

Reason and Apply

6. There are approximately 5.58×10^{21} atoms in a gram of silver. How many atoms are there in 3 kilograms of silver? Express your answer in scientific notation. @ 1.674×10^{25}

7. Because the number of molecules in a given amount of a compound is usually a very large number, scientists often work with a quantity called a *mole*. One mole is about 6.02×10^{23} molecules.

a. A liter of water has about 55.5 moles of H_2O. How many molecules is this? Write your answer in scientific notation. 3.3411×10^{25}

b. How many molecules are in 6.02×10^{23} moles of a compound? Write your answer in scientific notation. approximately 3.6×10^{47}

The number of molecules in one mole is called *Avogadro's number*. The number is named after the Italian chemist and physicist Amadeo Avogadro (1776–1856).

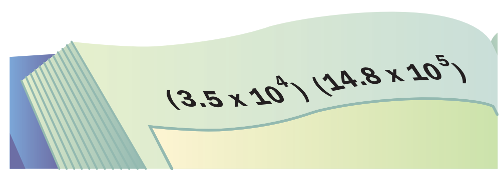
8. Write each number in scientific notation. How does your calculator show each answer? Answers will vary based on the model of calculator used.

a. 250 2.5×10^2; 2.5E2

b. 7,420,000,000,000 7.42×10^{12}; 7.42E12

c. -18 -1.8×10^1; -1.8E1

9. Cal and Al were assigned this multiplication problem for homework:



Cal got an answer of 51.8×10^9, and Al got 5.18×10^{10}.

a. Are Cal's and Al's answers equivalent? Explain why or why not. @

b. Whose answer is in scientific notation? @ Al's answer

c. Find another exponential expression equivalent to Cal's and Al's answers. @ possible answer: 518×10^8

d. Explain how you can rewrite a number such as 432.5×10^3 in scientific notation. @

10. Consider these multiplication expressions:

i. $(2 \times 10^5)(3 \times 10^8)$ 6×10^{13} **ii.** $(6.5 \times 10^3)(2.0 \times 10^5)$ 1.3×10^9

a. Set your calculator in scientific notation mode and multiply each expression.

b. Explain how you could do the multiplication in 10a without using a calculator. ⓗ

c. Find the product $(4 \times 10^5)(6 \times 10^7)$ and write it in scientific notation without using your calculator.
$(4 \times 10^5)(6 \times 10^7) = 4 \times 6 \times 10^5 \times 10^7 = 24 \times 10^{12} = 2.4 \times 10^1 \times 10^{12} = 2.4 \times 10^{13}$

11. Americans make almost 2 billion telephone calls each day. (*www.britannica.com*)

a. Write this number in standard notation and in scientific notation. 2,000,000,000; 2×10^9

b. How many phone calls do Americans make in one year? (Assume that there are 365 days in a year.) Write your answer in scientific notation. 7.3×10^{11} calls per year

12. On average a person sheds 1 million dead skin cells every 40 minutes. (*The World in One Day,* 1997, p. 16)

a. How many dead skin cells does a person shed in an hour? Write your answer in scientific notation. ⓗ

b. How many dead skin cells does a person shed in a year? (Assume that there are 365 days in a year.) Write your answer in scientific notation.
1.314×10^{10} cells per year

Dead skin cells are one of the components of dust.

13. A *light-year* is the distance light can travel in one year. This distance is approximately 9,460 billion kilometers. The Milky Way galaxy is estimated to be about 100,000 light-years in diameter.

a. Write both distances in scientific notation. 9.46×10^{12} km; 1.0×10^{5} light-years

b. Find the diameter of the Milky Way in kilometers. Use scientific notation. 9.46×10^{17} km

c. Scientists estimate the diameter of Earth is greater than 1.27×10^4 km. How many times larger is the diameter of the Milky Way? $\dfrac{(9.46 \times 10^{17})}{(1.27 \times 10^{4})} = 7.45 \times 10^{13}$

Review

6.2 **14.** **APPLICATION** The exponential equation $P = 3.8(1 + 0.017)^t$ approximates Australia's annual population (in millions) since 1900.

a. Explain the real-world meaning of each number and variable in the equation. ⓐ

b. What interval of t-values will give information up to the current year? ⓐ

c. Graph $P = 3.8(1 + 0.017)^t$ over the time interval you named in 14b.

d. What population does the model predict for the year 1950? ⓐ approximately 8.8 million

e. Use the equation to predict today's population. ⓗ
Answers will vary depending on the current year; $P = 3.8(1 + 0.017)^{current\ year - 1900}$.

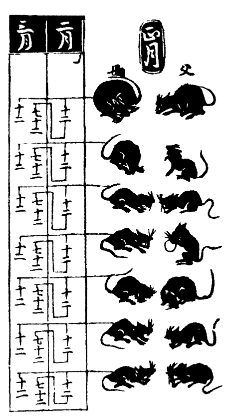
5.6 **15.** Graph $y \leq -2(x - 5)$.

IMPROVING YOUR REASONING SKILLS

The *Jinkōki* (Wasan Institute, 2000, p. 146) tells this ancient Japanese problem:

> A breeding pair of rats produced 12 baby rats (6 female and 6 male) in January. There were 14 rats at that time. In February, each female-male pair of rats again bred 12 baby rats. The total number of rats was then 98. In this way, each month, the parents, their children, their grandchildren, and so forth, breed 12 baby rats each. How many rats would there be at the end of one year?

Solve this problem using an exponential model. If you use your calculator, you will get an answer in scientific notation that doesn't show all the digits of the answer. Devise a way to find the "missing" digits.



Exercises 12 and 13 Encourage the use of dimensional analysis.

12a. 1.5×10^6 cells per hour

Exercise 13 This exercise, which reviews Lesson 6.3, can be used to foreshadow division with exponents in Lesson 6.5. That is, $10^{17} \div 10^4 = 10^{13}$.

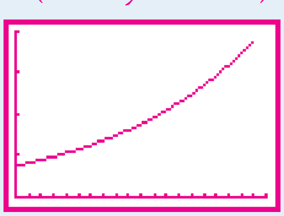
Exercise 14 Because the equation was written to fit data, it might not provide the exact population at any point. In particular, it predicts today's population a little higher than the actual population now.

This exercise is from *Graphic Algebra* (Key Curriculum Press), a good resource for other problems.

14a. 3.8 is the population (in millions) in 1900; 0.017 is the annual growth rate; t is the elapsed time in years since 1900; P is the population (in millions) t years after 1900.

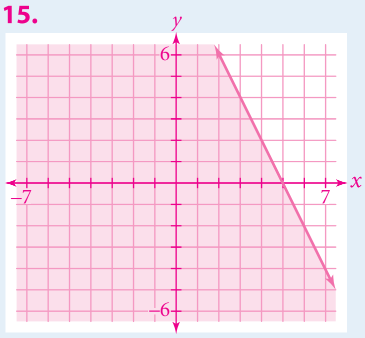
14b. Answers will vary depending on the current year; $0 \leq t \leq (current\ year - 1900)$.

14c.



[0, 100, 5, 0, 20, 5]

15.



IMPROVING REASONING SKILLS

One way to solve this problem is to concentrate on pairs of rats rather than individual rats. Every month, each pair produces 6 more pairs, making a total of 7 pairs at the end of the month for each pair at the beginning of the month. Therefore, there's a constant multiplier of 7, and an exponential model for the number of pairs is 7^x. Doubling 7^{12}, to find the number of individual rats after 12 months, gives 27,682,574,402. On the TI-83 Plus, the number is shown in scientific notation as 2.76825744E10—the last two digits are not displayed. If students subtract 27,000,000,000, the result is 682,574,402 and the final two digits are revealed. (Instead of entering 27000000000 to subtract, they can enter 2.7*10^10.)

LESSON

6.5

PLANNING

LESSON OUTLINE

One day:

- 20 min Investigation
- 5 min Sharing
- 10 min Examples
- 5 min Closing
- 10 min Exercises

MATERIALS

- Properties of Exponents (T), *optional*

TEACHING

Real-world situations give insight into why exponents are subtracted when powers of like bases are divided.

Guiding the Investigation

One Step

Pose this problem: "Six years ago Anne paid $18,500 for a van for her flower delivery service. Its value has been depreciating at a rate of 9% per year, so the van is currently worth $18{,}500(1-0.09)^6$ dollars. How much was it worth two years ago?" As you circulate, encourage students to write the value in at least two different ways: $\frac{18{,}500(1-0.09)^6}{(1-0.09)^2}$ and $18{,}500(1-0.09)^4$. Ask students if they think they can always subtract exponents when dividing, bring out the idea of having like bases, and ask them to make up and solve some related problems with which to challenge each other.

See page 724 for answers to Step 1.

LESSON

6.5

Looking Back with Exponents

The eye that directs a needle in the delicate meshes of embroidery, will equally well bisect a star with the spider web of the micrometer.

MARIA MITCHELL

You've learned that looking ahead in time to predict future growth with an exponential model is related to the multiplication property of exponents. In this lesson you'll discover a rule for dividing expressions with exponents. Then you'll see how dividing expressions with exponents is like looking *back* in time.

Investigation

The Division Property of Exponents

Step 1 Write the numerator and the denominator of each quotient in expanded form. Then reduce to eliminate common factors. Rewrite the factors that remain with exponents. Use your calculator to check your answers.

a. $\frac{5^9}{5^6}$ **b.** $\frac{3^3 \cdot 5^3}{3 \cdot 5^2}$ **c.** $\frac{4^4x^6}{4^2x^3}$

Step 2 Compare the exponents in each final expression you got in Step 1 to the exponents in the original quotient. Describe a way to find the exponents in the final expression without using expanded form.

Step 2 Descriptions should include subtracting the exponent in the denominator from the exponents in the numerator.

Step 3 Use your method from Step 2 to rewrite this expression so that it is not a fraction. You can leave $\frac{0.08}{12}$ as a fraction.

$$\frac{5^{15}\left(1+\frac{0.08}{12}\right)^{24}}{5^{11}\left(1+\frac{0.08}{12}\right)^{18}} \quad 5^{(15-11)}\left(1+\frac{0.08}{12}\right)^{(24-18)} = 5^4\left(1+\frac{0.08}{12}\right)^6$$

Recall that exponential growth is related to repeated multiplication. When you look ahead in time you multiply by repeated constant multipliers, or increase the exponent. To look back in time you will need to undo some of the constant multipliers, or divide.

LESSON OBJECTIVE

- Review or learn the division property of exponents

NCTM STANDARDS

CONTENT		PROCESS	
✓	Number	✓	Problem Solving
✓	Algebra	✓	Reasoning
	Geometry	✓	Communication
	Measurement	✓	Connections
	Data/Probability	✓	Representation

Step 4 Apply what you have discovered about dividing expressions with exponents.

a. After 7 years the balance in a savings account is $500(1+0.04)^7$. What does the expression $\frac{500(1+0.04)^7}{(1+0.04)^3}$ mean in this situation? Rewrite this expression with a single exponent. the balance 3 yr prior; $500(1+0.04)^4$

b. After 9 years of depreciation, the value of a car is $21{,}300(1-0.12)^9$. What does the expression $\frac{21{,}300(1-0.12)^9}{(1-0.12)^5}$ mean in this situation? Rewrite this expression with a single exponent. the balance 5 yr prior; $21{,}300(1-0.12)^4$

c. After 5 weeks the population of a bug colony is $32(1+0.50)^5$. Write a division expression to show the population 2 weeks earlier. Rewrite your expression with a single exponent. $\frac{32(1+0.50)^5}{(1+0.50)^2}=32(1+0.50)^3$

d. The expression $A(1+r)^n$ can model n time periods of exponential growth. What expression models the growth m time periods earlier? $A(1+r)^{n-m}$

Step 5 How does looking back in time with an exponential model relate to dividing expressions with exponents? Dividing by $(1+r)^m$ represents looking back m time periods.

Expanded form helps you understand many properties of exponents. It also helps you understand how the properties work together.

EXAMPLE A

Use the properties of exponents to rewrite each expression.

a. $\frac{6x^9}{5x^4}$ **b.** $\frac{(3x^2)(8x^4)}{-4x^3}$ **c.** $\frac{7.5\times10^8}{1.5\times10^3}$

▶ Solution

a.

Use expanded form and reduce.

$$\frac{6x^9}{5x^4}=\frac{6\cdot x\cdot x\cdot x\cdot x\cdot x\cdot \cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}}{5\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}}=\frac{6x^5}{5}\text{, or }1.2x^5$$

$$\frac{6x^9}{5x^4}=\frac{6x^{9-4}}{5}=\frac{6x^5}{5}\text{, or }1.2x^5$$

In expanded form, 4 factors of x are removed in the numerator and denominator. That leaves $9-4$, or 5, factors of x in the numerator.

b.

Use expanded form and reduce.

$$\frac{(3x^2)(8x^4)}{-4x^3}=\frac{3\cdot8\cdot x^2\cdot x^4}{-4\cdot x^3}=\frac{3\cdot8}{-4}\cdot\frac{x\cdot x\cdot x\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}}{\cancel{x}\cdot\cancel{x}\cdot\cancel{x}}=-6x^3$$

$$\frac{(3x^2)(8x^4)}{-4x^3}=\frac{3\cdot8\cdot x^2\cdot x^4}{-4\cdot x^3}=\frac{3\cdot8}{-4}\cdot x^{(2+4)-3}=-6x^3$$

In expanded form, 2 factors of x are combined with 4 factors of x in the numerator. Then 3 factors of x are removed in the numerator and denominator. That leaves $(2+4)-3$, or 3, factors of x in the numerator.

Step 1 Encourage students to write exponents of 1. Doing so will aid them in devising a rule in Step 2.

Step 3 Division by a power of 5 as well as the other power might confuse some students. Remind them that division is the opposite of multiplication, so that they can think of dividing 5^{15} by 5^{11}.

Step 4 If students are having difficulty expressing their ideas in parts a and b, suggest that they look at the wording in parts c and d.

SHARING IDEAS

Ask students to share their ideas about Step 2. Try to get them to formulate the division property of exponents. Students can look on page 362, or you might want to display the division property shown on the Properties of Exponents transparency.

[Ask] "Why must b not equal zero?" [Division by zero is undefined.] "Why doesn't the statement exclude m or n from being zero?" Getting students to conjecture about the value of b^0 motivates Lesson 6.6.

Assessing Progress

Look for understanding that exponents represent repeated multiplication, understanding that the fraction bar represents division, and the ability to see "chunks."

▶ EXAMPLE A

This example is good for extending the division property beyond powers on a single base. If students are having trouble, keep stressing that they can write out the terms in expanded form, such as

$$\frac{6x^9}{5x^4}=\frac{6\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x}{5\cdot x\cdot x\cdot x\cdot x},$$

and then eliminate factors equivalent to 1, in this case getting $\frac{6x^5}{5}$.

c.

$$\frac{7.5 \times 10^8}{1.5 \times 10^3} = \frac{7.5}{1.5} \times \frac{10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot \cancel{10} \cdot \cancel{10} \cdot \cancel{10}}{\cancel{10} \cdot \cancel{10} \cdot \cancel{10}} = 5.0 \times 10^5$$

$$\frac{7.5}{1.5} \times 10^{8-3}$$

So, division involving scientific notation can be done just like any other expression with exponents.

The investigation and example have introduced the **division property of exponents.**

Division Property of Exponents

For any nonzero value of b and any integer values of m and n,

$$\frac{b^n}{b^m} = b^{n-m}$$

The division property of exponents lets you divide expressions with exponents simply by subtracting the exponents.

EXAMPLE B

Six years ago, Anne bought a van for \$18,500 for her flower delivery service. Based on the prices of similar used vans, she estimates a rate of depreciation of 9% per year.

a. How much is the van worth now?

b. How much was it worth last year?

c. How much was it worth 2 years ago?

▶ Solution

The original price was \$18,500, and the rate of depreciation as a decimal is 0.09. Use the expression $A(1-r)^x$.

a. Right now the value of the van has been decreasing for 6 years.

$$A(1-r)^x = 18{,}500(1-0.09)^6 \approx 10{,}505.58$$

The van is currently worth \$10,505.58.

b. A year ago, the van was 5 years old. One approach is to use 5 as the exponent.

$$18{,}500(1-0.09)^5 \approx 11{,}544.59$$

Another approach is to undo the multiplication in part a by using division.

$$\frac{18{,}500(1-0.09)^6}{(1-0.09)} = 18{,}500(1-0.09)^5$$

The numerator on the left side of this equation represents the starting value multiplied by 6 factors of the constant multiplier $(1-0.09)$. Dividing by the constant multiplier once leaves you with an expression representing 5 years of exponential depreciation. Either way, the exponent is decreased by 1. The van was worth \$11,544.59 last year.

If b is positive, the division property of exponents works for all real values of m and n.

▶ *EXAMPLE B*

This example provides another case of a decreasing exponential situation, especially useful for students who had difficulties with Step 4b of the investigation.

Closing the Lesson

Considering the value of an exponentially growing quantity at one time and at an earlier time can show that, when powers of like bases are divided, the exponents are subtracted.

c. To find the value 2 years ago, decrease the exponent in part a by 2.

$$18{,}500(1 - 0.09)^{6-2} = 18{,}500(1 - 0.09)^4 \approx 12{,}686.37$$

Subtracting 2 from the exponent gives the same result as undoing two multiplications. The van was worth \$12,686.37 two years ago.

EXERCISES

Practice Your Skills

1. Eliminate factors equivalent to 1 and rewrite the right side of this equation.

$$\frac{x^5y^4}{x^2y^3} = \frac{x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y}{x \cdot x \cdot y \cdot y \cdot y} \quad x^3y$$

2. Use the properties of exponents to rewrite each expression.

a. $\frac{7^{12}}{7^4}$ ⓐ 7^8 **b.** $\frac{x^{11}}{x^5}$ x^6 **c.** $\frac{12x^5}{3x^2}$ ⓐ $4x^3$ **d.** $\frac{7x^6y^3}{14x^3y}$ $0.5x^3y^2$

3. Cal says that $\frac{3^6}{3^2}$ equals 1^4 because you divide the 3's and subtract the exponents. Al knows Cal is incorrect, but he doesn't know how to explain it. Write an explanation so that Cal will understand why he is wrong and how he can get the correct answer. ⓗ

4. **APPLICATION** Webster owns a set of antique dining-room furniture that has been in his family for many years. The historical society tells him that furniture similar to his has been appreciating in value at 10% per year for the last 20 years and that his furniture could be worth \$10,000 now.

a. Which letter in the equation $y = A(1 + r)^x$ could represent the value of the furniture 20 years ago when it started appreciating? ⓐ

b. Substitute the other given information into the equation $y = A(1 + r)^x$. ⓐ

c. Solve your equation in 4b to find how much Webster's furniture was worth 20 years ago. Show your work. ⓐ

5. Use the properties of exponents to rewrite each expression.

a. $(2x)^3 \cdot (3x^2)^4$ $648x^{11}$ **b.** $\frac{(5x)^7}{(5x)^5}$ $25x^2$ **c.** $\frac{(2x)^5}{-8x^3}$ ⓐ $-4x^2$ **d.** $(4x^2y^5) \cdot (-3xy^3)^3$ $-108x^5y^{14}$

Reason and Apply

6. Earth is 1.5×10^{11} m from the Sun. Light travels at a speed of 3×10^8 m/s. How long does it take light to travel from the Sun to Earth? Answer to the nearest minute.
It takes 500 s, or approximately 8 min.

BUILDING UNDERSTANDING

Students practice subtracting exponents when dividing powers of like bases.

ASSIGNING HOMEWORK

Essential	1–8
Performance assessment	4, 9, 11–13
Portfolio	8, 10
Journal	3
Group	10
Review	14, 15

Helping with the Exercises

Exercise 2a If students write this as 1^8, urge them to write out the 7's and then remove values of 1. There will be eight 7's left.

Exercise 3 This exercise gives students a chance to confront a common error. **[Ask]** "Why can't you divide the common bases?" Encourage students to write the numerator and denominator in expanded form and then remove values equivalent to 1. To challenge students who understand this very well, **[Ask]** "Are there any conditions under which you could divide the common bases?" [If the same bases have the same power, then the ratio is 1. Dividing the bases to get a base of 1 will yield that ratio.]

3. Possible answer: $\frac{3^6}{3^2}$ means there are six factors of 3 in the numerator and two factors of 3 in the denominator. So there are two factors of 1, or $\frac{3}{3}$, in the entire expression, leaving four factors of 3 in the numerator, or 3^4.

Exercise 4 At this point students can solve the equation by using graphing or tables. Students will see in Lesson 6.6 that the equation can also be solved using negative exponents: $A = 10{,}000(1 + 0.1)^{-20}$.

4a. A represents the starting value.

4b. $10{,}000 = A(1 + 0.1)^{20}$

4c. $10{,}000 = A(1 + 0.1)^{20}$

$$\frac{10{,}000}{(1 + 0.1)^{20}} = A$$

$$1486.43 \approx A$$

The furniture was worth about \$1,486 twenty years ago.

Exercises 6 and 7 Encourage the use of dimensional analysis in setting up the quotient. If students use a calculator to do the calculations, point out that they can also do some of the calculations using the division property of exponents.

7. **APPLICATION** **Population density** is the number of people per square mile. That is, if the population of a country were spread out evenly across an entire nation, the population density would be the number of people in each square mile.

a. In 2004, the population of Mexico was about 1.0×10^8. Mexico has a land area of about 7.6×10^5 square miles. What was the population density of Mexico in 2004? (Central Intelligence Agency, *www.cia.gov*) @ about 132 people per square mile

b. In 2004, the population of Japan was about 1.3×10^8. Japan has a land area of about 1.5×10^5 square miles. What was the population density of Japan in 2004? (Central Intelligence Agency, *www.cia.gov*) about 867 people per square mile

c. How did the population densities of Mexico and Japan compare in 2004?

7c. The population of Japan was about 6.6 times denser than that of Mexico.

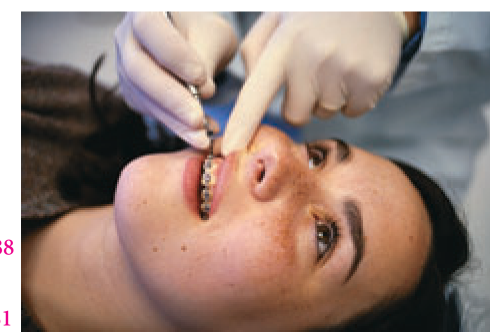
8. **APPLICATION** Eight months ago, Tori's parents put \$5,000 into a savings account that earns 3% annual interest. Now, her dentist has suggested that she get braces.

a. If the interest is calculated each month, what is the monthly interest rate? @ 0.25%

b. If Tori's parents use the money in their savings account, how much do they have? \$5,100.88

c. If Tori's dentist had suggested braces 3 months ago, how much money would have been in her parents' savings account? \$5,062.81

d. Tori's dentist says she can probably wait up to 2 months before having the braces fitted. How much will be in her parents' savings account if she waits? \$5,126.42



Orthodontic treatment can cost between \$4,000 and \$6,000 depending on the extent of the procedure. An estimated 5 million people were treated by orthodontists in the United States in 2000.

9. **APPLICATION** During its early stages, a disease can spread exponentially as those already infected come in contact with others. Assume that the number of people infected by a disease approximately triples every day. At one point in time, 864 people are infected. How many days earlier had fewer than 20 people been infected? Show two different methods for solving this problem. ⓗ

Exercise 9 Students could also use the equation $20 = 864(3)^x$, thus previewing negative exponents in Lesson 6.6.

9. Four days earlier;

Method 1: Use a recursive routine: 864 ENTER, Ans/3 ENTER, ENTER, ENTER, ENTER.

Method 2: Use an equation: $y = 864\left(\frac{1}{3}\right)^x$; look at the table to find x when y is less than 20.

10. The population of a city has been growing at a rate of 2% for the last 5 years. The population is now 120,000. Find the population 5 years ago. approximately 108,688

11. **APPLICATION** In the course of a mammal's lifetime, its heart beats about 800 million times, regardless of the mammal's size or weight. (This excludes humans.)

a. An elephant's heart beats approximately 25 times a minute. How many years would you expect an elephant to live? Use scientific notation to calculate your answer. @ **approximately 61 yr**

b. A pygmy shrew's heart beats approximately 1150 times a minute. How many years would you expect a pygmy shrew to live? **approximately 1.3 yr**

c. If this relationship were true for humans, how many years would you expect a human being with a heart rate of 60 bpm to live? **approximately 25.4 yr**

Pygmy shrews may be the world's smallest mammal, as small as 5 cm from nose to tail.

12. More than 57,000 tons of cotton are produced in the world each day. It takes about 8 ounces of cotton to make a T-shirt. The population of the United States in 2000 was estimated to be more than 275 million. If all the available cotton were used to make T-shirts, how many T-shirts could have been manufactured every day for each person in the United States in 2000? Write your answer in scientific notation. (*www.cotton.net*) **approximately 8.3×10^{-1} T-shirt per person**

13. Each day, bees sip the nectar from approximately 3 trillion flowers to make 3300 tons of honey. How many flowers does it take to make 8 ounces of honey? Write your answer in scientific notation. (*The World in One Day,* 1997, p. 21) ⓗ **approximately 2.272×10^5 flowers**

Review

2.3 **14.** On his birthday Jon figured out that he was 441,504,000 seconds old. Find Jon's age in years. (Assume that there are 365 days per year.) **14 yr**

2.3, 6.4 **15.** Halley is doing a report on the solar system and wants to make models of the Sun and the planets showing relative size. She decides that Pluto, the smallest planet, should have a model diameter of 2 cm.

a. Using the table, find the diameters of the other models she would have to make. ⓗ

b. What advice would you give Halley on her project?

Possible answer: Halley should make her models much smaller because Jupiter is 1 m in diameter and the Sun is greater than 11 m in diameter; it would be better to leave the Sun out of her models altogether. If she makes Pluto with a diameter of 0.2 cm, Jupiter will be only about 12 cm in diameter.

Size of Planets and Sun

Planet	Diameter (mi)
Mercury	3.1×10^3
Venus	7.5×10^3
Earth	7.9×10^3
Mars	4.2×10^3
Jupiter	8.8×10^4
Saturn	7.1×10^4
Uranus	5.2×10^4
Neptune	3.1×10^4
Pluto	1.5×10^3
Sun	8.64×10^5

Exercise 11 Encourage students to use dimensional analysis. As needed, remind them that bpm means *beats per minute.*

Exercises 12 and 13 Urge students to use dimensional analysis carefully. Students may not know that 1 ton is 2000 lb and 1 lb is 16 oz.

Exercise 15 This exercise requires division of exponential expressions.

15a. **(answers recorded to tenths) Mercury: 4.1 cm; Venus: 10 cm; Earth: 10.5 cm; Mars: 5.6 cm; Jupiter: 117.3 cm; Saturn: 94.7 cm; Uranus: 69.3 cm; Neptune: 41.3 cm; Pluto: 2 cm; Sun: 1152 cm**

LESSON

6.6

PLANNING

LESSON OUTLINE

First day:

40 min Investigation

10 min Sharing

Second day:

30 min Examples

5 min Closing

15 min Exercises

MATERIALS

- Properties of Exponents (T), *optional*

TEACHING

Negative exponents are used in scientific notation for numbers very close to zero. They provide an alternative way to solve problems involving division of powers.

Guiding the Investigation

One Step

Pose this problem: "Assume that the diameter of a pin point is 0.0010 cm and that each atom has diameter 0.00000001 cm. How would you represent these numbers with scientific notation, and how many atoms are on the point of a pin?" As students work, remind them that the positive exponent in scientific notation is one fewer than the number of digits to the left of the decimal point. After they get the quotient $\frac{1.0 \times 10^{-3}}{10^{-8}}$, you may need to remind them how to subtract a negative number from another negative number.

Step 4 Possible answer: A base raised to a negative exponent means the same thing as the reciprocal of the base raised to the same exponent with the sign changed to positive.

LESSON

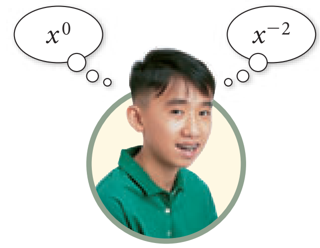
6.6

Zero and Negative Exponents

It is not knowledge which is dangerous, but the poor use of it.

HROTSWITHA

Have you noticed that so far in this chapter the exponents have been positive integers? In this lesson you will learn what a zero or a negative integer means as an exponent.



Investigation
More Exponents

Step 1 Use the division property of exponents to rewrite each of these expressions with a single exponent. Use your calculator to check your answers.

a. $\frac{y^7}{y^2}$ y^5 **b.** $\frac{3^2}{3^4}$ 3^{-2} **c.** $\frac{7^4}{7^4}$ 7^0 **d.** $\frac{2}{2^5}$ 2^{-4} **e.** $\frac{x^3}{x^6}$ x^{-3}

f. $\frac{z^8}{z}$ z^7 **g.** $\frac{2^3}{2^3}$ 2^0 **h.** $\frac{x^5}{x^5}$ x^0 **i.** $\frac{m^6}{m^3}$ m^3 **j.** $\frac{5^3}{5^5}$ 5^{-2}

Some of your answers in Step 1 should have positive exponents, some should have negative exponents, and some should have a zero exponent.

Step 2 Possible answer: Expressions resulting in positive exponents have a larger exponent in the numerator; expressions resulting in negative exponents have larger exponents in the denominator; expressions resulting in zero exponents have equal exponents in the numerator and denominator.

Step 2 How can you tell what type of exponent will result simply by looking at the original expression?

Step 3 Go back to the expressions in Step 1 that resulted in a negative exponent. Write each in expanded form. Then reduce them. **b.** $\frac{1}{9}$ **d.** $\frac{1}{16}$ **e.** $\frac{1}{x^3}$ **j.** $\frac{1}{25}$

Step 4 Compare your answers from Step 3 and Step 1. Tell what a base raised to a negative exponent means.

Step 5 Go back to the expressions in Step 1 that resulted in an exponent of zero. Write each in expanded form. Then reduce them. c, g, and h; 1

Step 6 Compare your answers from Step 5 and Step 1. Tell what a base raised to an exponent of zero means.
Possible answer: A base raised to a zero exponent is always equal to 1.

LESSON OBJECTIVES

- Investigate the meaning of nonpositive exponents
- Write a number with a negative exponent in a form that has a positive exponent and write a number with a positive exponent in a form that has a negative exponent
- Write in scientific notation numbers close to zero

NCTM STANDARDS

	CONTENT		PROCESS
✓	Number		Problem Solving
✓	Algebra	✓	Reasoning
	Geometry	✓	Communication
✓	Measurement	✓	Connections
	Data/Probability	✓	Representation

Step 7 Use what you have learned about negative exponents to rewrite each of these expressions with positive exponents and only one fraction bar.

a. $\frac{5^{-2}}{1}$ $\frac{1}{5^2}$

b. $\frac{1}{3^{-8}}$ $\frac{1}{\frac{1}{3^8}}$, or $\frac{3^8}{1}$

c. $\frac{4x^{-2}}{z^2y^{-5}}$ $\frac{4y^5}{z^2x^2}$

Step 8 In one or two sentences, explain how to rewrite a fraction with a negative exponent in the numerator or denominator as a fraction with positive exponents. **Possible answer: An expression with an exponent can be moved between the numerator and denominator of a fraction as long as the sign of the exponent is changed with each move.**

This table supports what you have learned about negative exponents and zero exponents. To go down either column of the table, you divide by 3. Notice that each time you divide, the exponent decreases by 1. (Likewise, to go up either column of the table, you multiply by 3 and the exponent increases by 1.) In order to continue the pattern, 3^0 must have the value 1. As the exponents become negative, the base 3 appears in the denominator with a positive exponent.

Exponential form	Fraction form
3^3	27
3^2	9
3^1	3
3^0	1
3^{-1}	$\frac{1}{3}$
3^{-2}	$\frac{1}{9}$
3^{-3}	$\frac{1}{27}$

$3^1 \div 3 = \frac{3^1}{3^1} = 3^{1-1} = 3^0$

$3 \div 3 = \frac{3}{3} = 1$

$3^{-1} \div 3 = \frac{3^{-1}}{3^1} = 3^{-1-1} = 3^{-2}$

$\frac{1}{3} \div 3 = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{3^2} = \frac{1}{9}$

Negative Exponents and Zero Exponents

For any nonzero value of b and for any value of n,

$$b^{-n} = \frac{1}{b^n} \quad \text{and} \quad \frac{1}{b^{-n}} = b^n$$

$$b^0 = 1$$

EXAMPLE A Use the properties of exponents to rewrite each expression without a fraction bar.

a. $\frac{3^5}{4^7}$

b. $\frac{25}{x^8}$

c. $\frac{5^{-3}}{2^{-8}}$

d. $\frac{3(17)^8}{17^8}$

Step 7 Some students may see that a negative exponent on a term in the numerator moves that term to the denominator with a positive exponent but not see that the process works in reverse. Suggest that they write the denominator in part b as a fraction and then multiply the numerator and denominator of the complex fraction by 3^8. The same idea applies to y^{-5} in part c.

Step 8 **[Ask]** "Do your findings extend to rewriting an expression with positive exponents as a fraction with negative exponents?"

SHARING IDEAS

Ask for ideas from Step 8. Work with the class to come up with a precise method. **[Ask]** "Why does the method work?" [One justification might be through patterns, as found in the table in the student text.]

Have the class critique the boxed definition in the student text. Ask why b must be "nonzero." Help students see that if b were zero, then you'd have $0 = \frac{1}{0^n}$, but division by zero is undefined. When students realize the importance of the phrase "for any value of n," they may raise questions about the case of $n = 0$. In that case, both sides of the equation are 1.

Elicit the question of the value of 0^0. Arguments for two different values can be made: zero to any other power is zero, but any other number to the zero power is one. Mathematicians say 0^0 is *indeterminate,* meaning that it's undefined. If students are interested, they might graph the equation $y = x^x$ on their calculators and see what value it gets close to as x approaches zero.

Note the quotation introducing the lesson. Hrotswitha of Gansersheim (935–1000 C.E.) is the first recorded female German writer. She included some mathematics in her plays. She was also a Benedictine nun. You could use this quote to warn against the "dangers" of misapplying the various properties of exponents, though she undoubtedly had in mind a deeper social meaning.

Assessing Progress

You can assess students' understanding of exponents as indicators of repeated multiplication, their understanding of subtraction of exponents when dividing like bases, and their ability to subtract a larger integer from a smaller one.

▶ ***EXAMPLE A***

This example is useful for students who didn't understand the rule derived in the investigation, especially the case of a negative exponent in the denominator.

▶ ***EXAMPLE B***

This example shows how negative exponents can be applied to a type of problem introduced in the previous lesson.

You may want to show that the solution to the equation $5600 = A(1 - 0.15)^3$ is equivalent to $A = \frac{5600}{(1-0.15)^3}$, or $5600(1 - 0.15)^{-3}$.

When time (in the exponent) is negative, the reference is to the time before some fixed point in time, usually the present. You might extend the example by asking students to use negative exponents to rework some exercises from Lesson 6.5.

▶ **Solution**

a. $\frac{3^5}{4^7} = 3^5 \cdot \frac{1}{4^7}$ — Think of the original expression as having two separate factors.

$= 3^5 \cdot 4^{-7}$ — Use the definition of negative exponents.

b. $\frac{25}{x^8} = 25 \cdot \frac{1}{x^8} = 25 \cdot x^{-8} = 25x^{-8}$

c. $\frac{5^{-3}}{2^{-8}} = 5^{-3} \cdot \frac{1}{2^{-8}} = 5^{-3} \cdot 2^8$

d. $\frac{3(17)^8}{17^8} = 3 \cdot 17^0$ — Use the division property of exponents.

$= 3 \cdot 1$ — Use the definition of zero exponents.

$= 3$ — Multiply.

You can also use negative exponents to look back in time with increasing or decreasing exponential situations.

EXAMPLE B

Solomon bought a used car for \$5,600. He estimates that it has been decreasing in value by 15% each year.

a. If his estimate of the rate of depreciation is correct, how much was the car worth 3 years ago?

b. If the car is 7 years old, what was the original price of the car?

▶ **Solution**

a. You can solve this problem by considering \$5,600 to be the starting value and then looking back 3 years.

$y = A(1 - r)^x$ — The general form of the equation.

$y = 5{,}600(1 - 0.15)^{-3}$ — Substitute the given information in the equation. -3 means you look back 3 years.

$y \approx 9{,}118.66$

The value of the car 3 years ago was approximately \$9,118.66.

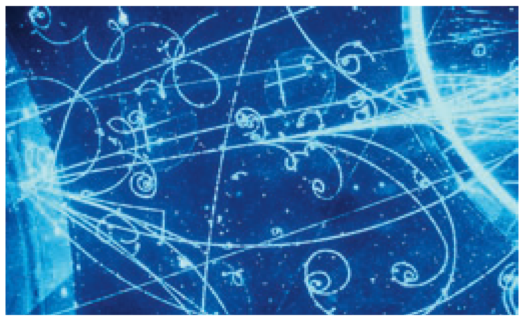
b. The original price is the value of the car 7 years ago.

$$y = 5{,}600(1 - 0.15)^{-7}$$
$$y \approx 17{,}468.50$$

The original price was approximately $17,468.50.

You can also use negative exponents to write numbers close to 0 in scientific notation. Just as positive powers of 10 help you rewrite numbers with lots of zeros, negative powers of 10 help you rewrite numbers with lots of zeros between the decimal point and a nonzero digit.

EXAMPLE C



These particle tracks show the paths of particles like protons, electrons, and mesons during a nuclear reaction.

Convert each number to standard notation from scientific notation, or vice versa.

a. A pi meson, an unstable particle released in a nuclear reaction, "lives" only 0.000000026 s.

b. The number 6.67×10^{-11} is the gravitational constant in the metric system used to calculate the gravitational attraction between two objects that have given masses and are a given distance apart.

c. The mass of an electron is 9.1×10^{-31} kg.

▸ Solution

a. $0.000000026 = \dfrac{2.6}{100{,}000{,}000} = \dfrac{2.6}{10^8} = 2.6 \times 10^{-8}$

Notice that the decimal point in the original number was moved to the right eight places to get a number between 1 and 10, in this case, 2.6. To undo that, you must multiply 2.6 by 10^{-8}.

b. $6.67 \times 10^{-11} = \dfrac{6.67}{10^{11}} = \dfrac{6.67}{100{,}000{,}000{,}000} = 0.0000000000667$

Multiplying 6.67 by 10^{-11} moves the decimal point 11 places to the left, requiring 10 zeros after the decimal point—the first move of the decimal point changes 6.67 to 0.667.

c. Generalize the method in part b. To write 9.1×10^{-31} in standard notation you move the decimal point 31 places to the left, requiring 30 zeros after the decimal point.

$$9.1 \times 10^{-31} = 0.00000000000000000000000000000091$$

▸ EXAMPLE C

[Language] The term *pi meson* is pronounced pie′-may′-zon.

The gravitational constant is popularly known as *G*. Newton's law of universal gravitation states that every particle in the universe attracts every other particle with a force directly proportional to the product of the masses of the two particles and inversely proportional to the square of the distance between their centers of mass. In symbols, $F = \dfrac{Gm_1m_2}{s^2}$, where *F* is the force of attraction in newtons, m_1 and m_2 are the masses in kilograms (kg), and *s* is the distance in meters (m). For units to cancel in dimensional analysis, *G* is in $\dfrac{\text{newton-m}^2}{\text{kg}^2}$.

Because the student text refers to moving decimal places, if you haven't done so already you may want to review the effects on the decimal point of multiplying and dividing by 10. Multiplying by 10 moves the decimal to the right one place, and dividing by 10 moves the decimal to the left one place. Some students remember this with a number line—negatives to the left, positives to the right.

Exponents can be decimals or other fractions as well as integers. **[Ask]** "What have you seen already in this chapter that indicates the existence of fractional and decimal exponents?" [The graphs of equations of the form $y = A(1 + r)^x$ are smooth curves, indicating that *x* can have non-integer values.]

Closing the Lesson

As needed, say that negative exponents are used in scientific notation for numbers very close to zero and provide an alternative way of solving problems involving division of powers.

BUILDING UNDERSTANDING

Students practice working with nonpositive exponents.

ASSIGNING HOMEWORK

Essential	1–4, 6, 7, 9
Performance assessment	8, 14
Portfolio	12
Journal	5, 14
Group	9, 10–13
Review	14, 15

Helping with the Exercises

Exercise 2 **[Alert]** This exercise tests students' number sense because they have to identify the order relationship between pairs of numbers. If students have difficulty with parts b and d, suggest that they treat the numbers as positive first and then multiply through by -1, changing the direction of the inequality.

Exercise 3b **[Alert]** Students need to be very careful: As they decrease 46 to 4.6, they add 1 to the negative exponent.

Exercise 4 Even students who would immediately recognize that $(1 + 0.028)^0$ equals 1 might be confused by needing to go backward.

Exercise 7 This exercise is about inflation, so the constant multiplier is more than 1, but the situation calls for thinking back in time, so the exponent is negative. **[Alert]** Students might write a negative quotient instead of using a negative exponent.

7a. $3500(1 + 0.04)^{-4}$; about \$2,992

7c. $25(1 + 0.04)^{-5}$; about \$21

EXERCISES

You will need your graphing calculator for Exercise **15.**

Practice Your Skills

1. Rewrite each expression using only positive exponents.

a. 2^{-3} @ $\frac{1}{2^3}$ **b.** 5^{-2} $\frac{1}{5^2}$ **c.** 1.35×10^{-4} @ $\frac{1.35}{10^4}$

2. Insert the appropriate symbol ($<$, $=$, or $>$) between each pair of numbers.

a. 6.35×10^5 $\boxed{=}$ 63.5×10^4 @ **b.** -5.24×10^{-7} $\boxed{<}$ -5.2×10^{-7}

c. 2.674×10^{-5} $\boxed{>}$ 2.674×10^{-6} **d.** -2.7×10^{-4} $\boxed{>}$ -2.8×10^{-3}

3. Find the exponent of 10 that you need to write each expression in scientific notation.

a. $0.0000412 = 4.12 \times 10^{\square}$ @ -5 **b.** $46 \times 10^{-5} = 4.6 \times 10^{\square}$ -4 **c.** $0.00046 = 4.6 \times 10^{\square}$ -4

4. The population of a town is currently 45,647. It has been growing at a rate of about 2.8% per year.

a. Write an expression in the form $45{,}647(1 + 0.028)^x$ for the current population. @ $45{,}647(1 + 0.028)^0$

b. What does the expression $45{,}647(1 + 0.028)^{-12}$ represent in this situation? @ the population 12 yr ago

c. Write and evaluate an expression for the population 8 years ago. @ $45{,}647(1 + 0.028)^{-8} \approx 36{,}599$

d. Write expressions without negative exponents that are equivalent to the exponential expressions from 4b and c. @ $\frac{45{,}647}{(1 + 0.028)^{12}}$; $\frac{45{,}647}{(1 + 0.028)^{8}}$

5. Juan says that 6^{-3} is the same as -6^3. Write an explanation of how Juan should interpret 6^{-3}, then show him how each expression results in a different value. Possible answer: Negative exponents mean to use a reciprocal base with the exponent positive; $6^{-3} = \frac{1}{6^3} = \frac{1}{216}$, $-6^3 = -216$.

Reason and Apply

6. Use the properties of exponents to rewrite each expression without negative exponents.

a. $(2x^3)^2(3x^4)$ $12x^{10}$ **b.** $(5x^4)^0(2x^2)$ $2x^2$

c. $3(2x)^3(3x)^{-2}$ @ $\frac{8x}{3}$ **d.** $\left(\frac{2x^4}{3x}\right)^{-3}$ $\frac{27}{8x^9}$

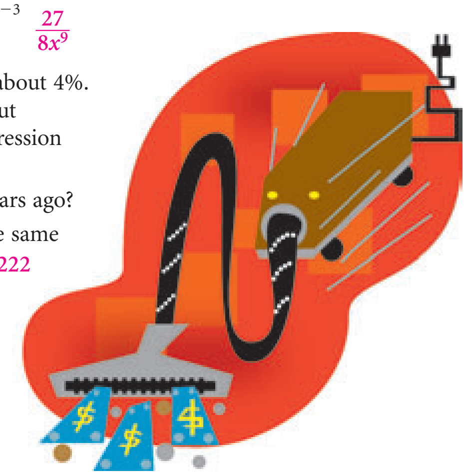
7. APPLICATION Suppose the annual rate of inflation is about 4%. This means that the cost of an item increases by about 4% each year. Write and evaluate an exponential expression to find the answers to these questions. ⓗ

a. If a piano costs \$3,500 today, what did it cost 4 years ago?

b. If a vacuum cleaner costs \$250 today, what did the same model cost 3 years ago? $250(1 + 0.04)^{-3}$; about \$222

c. If tickets to a college basketball game cost \$25 today, what did they cost 5 years ago?

d. The median price of a house in the United States in October 2004 was \$187,000. What was the median price 30 years ago? (National Association of Realtors, *www.realtor.org*) $187{,}000(1 + 0.04)^{-30}$; about \$57,656



8. **APPLICATION** The population of Japan in 2004 was about 1.3×10^8. Japan has a land area of about 1.5×10^5 square miles. (Central Intelligence Agency, *www.cia.gov*)

a. On average, how much land in square miles is there per person? (*Note:* This is a different problem from the one you may have solved in Lesson 6.5.) approximately 1.2×10^{-3} square mile per person

b. Convert your answer from 8a to square feet per person. approximately 3.35×10^4 square feet per person (or 3.22×10^4 square feet per person if the answer from 8a is used without rounding)

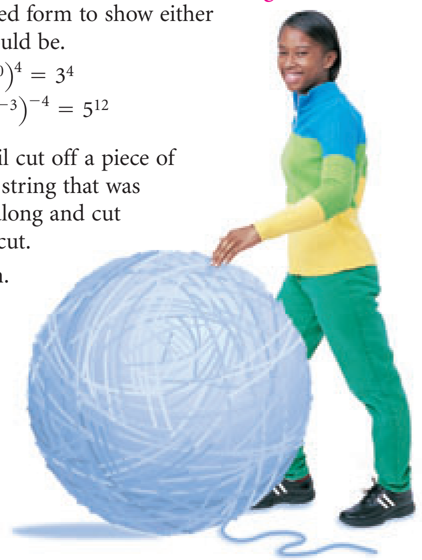
9. Decide whether each statement is true or false. Use expanded form to show either that the statement is true or what the correct statement should be.

a. $\left(2^3\right)^2 = 2^6$

b. $\left(3^0\right)^4 = 3^4$

c. $\left(10^{-2}\right)^4 = -10^8$ ⓐ

d. $\left(5^{-3}\right)^{-4} = 5^{12}$



10. A large ball of string originally held 1 mile of string. Abigail cut off a piece of string one-tenth of that length. Barbara then cut a piece of string that was one-tenth as long as the piece Abigail had cut. Cruz came along and cut a piece that was one-tenth the length of what Barbara had cut.

a. Write each length of string in miles in scientific notation.

b. If the process continues, how long a piece will the next person, Damien, cut off? 1×10^{-4} mi

c. Do any of the people have a piece of string too short to use as a shoelace? ⓗ Convert to inches; Damien's string is too short (6.3 in.).

11. Suppose $36(1 + 0.5)^4$ represents the number of bacteria cells in a sample after 4 hours of growth at a rate of 50% per hour. Write an exponential expression for the number of cells 6 hours earlier. $36(1 + 0.5)^{4-6}$, or $36(1 + 0.5)^{-2}$

12. **APPLICATION** Camila received a $1,200 prize for one of her essays. She decides to invest $1,000 of it for college. Her bank offers two options. The first is a regular savings account that pays 2.5% interest every 6 months. The second is a certificate of deposit that pays 5% interest each year.

a. With the savings account, how much would Camila have after 1 year? After 2 years? ⓐ $1,050.63; $1,103.81

b. With the certificate of deposit, how much would Camila have after 1 year? After 2 years? ⓐ $1,050; $1,102.50

c. Explain why you get different results for 12a and b. ⓐ Possible answer: In the savings account, interest is added at 6 mo, so the interest earns interest. The 1 yr interest is $(1 + 0.025)^2$, or 1.050625; that is more than 5%.

13. ***Mini-Investigation*** In the last few lessons, you have worked with equations that have a variable exponent, and you have dealt with positive, negative, and zero exponents. An equation in which *variables* are raised only to nonnegative integer exponents is called a **polynomial equation.** Identify these equations as exponential, polynomial, or neither. ⓗ

$y = 4x^3$ $\quad y = -3(1 + 0.4)^x$ $\quad y = x^x + x^2$ $\quad y = 2x^5 - 3x^2 + 4x + 2$

$y = 2 \cdot 3^x$ $\quad y = 2x + 7$ $\quad y = -6 + 2x + 3x^2$ $\quad y = 3$

Review

6.2 14. **APPLICATION** A capacitor is charged with a nine-volt battery. The equation $y = 9.4(1 - 0.043)^x$ models the charge of a capacitor after it is connected to a load. The variable x is in seconds since the capacitor is connected, and y is in volts.

a. Is the voltage of the capacitor increasing or decreasing? Explain. The voltage is decreasing because the base, 0.957, is less than 1.

10a. original: 1×10^0; Abigail: 1×10^{-1}; Barbara: 1×10^{-2}; Cruz: 1×10^{-3}

Exercise 13 The role of a constant is ambiguous. For example, the constant 3 could be thought of as $3x^0$ (a polynomial) or as $3 \cdot 0^x$ (an exponential). The convention is to consider a constant as a polynomial so that when it's added to a polynomial the result is still a polynomial. Polynomials will come up again in Chapter 9.

13. exponential: $y = -3(1 + 0.4)^x$, $y = 2 \cdot 3^x$; polynomial: $y = 4x^3$, $y = 2x^5 - 3x^2 + 4x + 2$, $y = 2x + 7$, $y = -6 + 2x + 3x^2$, $y = 3$; neither: $y = x^x + x^2$

9a. true; $\left(2^3\right)^2 = 2^3 \cdot 2^3 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6$

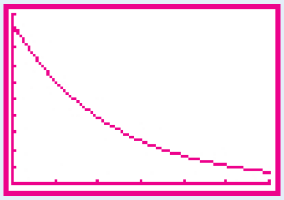
9b. false; $\left(3^0\right)^4 = (1)^4 = 1$

9c. false; $\left(10^{-2}\right)^4 = \left(\frac{1}{10^2}\right)^4 = \left(\frac{1}{10 \cdot 10}\right)\left(\frac{1}{10 \cdot 10}\right)\left(\frac{1}{10 \cdot 10}\right)\left(\frac{1}{10 \cdot 10}\right) = \frac{1}{10^8} = 10^{-8}$

9d. true; $\left(5^{-3}\right)^{-4} = \left(\frac{1}{5^3}\right)^{-4} = \frac{1}{\left(\frac{1}{5^3}\right)^4} = \frac{1}{\left(\frac{1}{5 \cdot 5 \cdot 5}\right)\left(\frac{1}{5 \cdot 5 \cdot 5}\right)\left(\frac{1}{5 \cdot 5 \cdot 5}\right)\left(\frac{1}{5 \cdot 5 \cdot 5}\right)} = \frac{1}{\frac{1}{5^{12}}} = \frac{1}{5^{-12}} = 5^{12}$

14b. 9.4 is the voltage at time 0. Each second, 4.3% of the previous voltage is lost.

14c.



[0, 60, 10, 0, 10, 1]

14d. When $x > 15.77$ s; possible answer: The exponential graph is below the graph of $y = 4.7$ for $x > 15.77$ s. (The exponential graph intersects the graph of $y = 4.7$ at $x \approx 15.7706$, but students will not solve to this precision.)

Exercise 15 **[Alert]** Students may forget to use parentheses when entering the denominators into their calculators. For example, in 15a, they may enter 8*10^8/2*10^3 instead of 8*10^8/(2*10^3). Remind them that the numerator is divided by each factor in the denominator. In fact, they can avoid parentheses by entering 8*10^8/2/10^3. Be sure they explain their reasoning in 15c.

15b. Possible answer: Divide the coefficients of the powers of 10, and then divide the powers of 10 (subtract the exponents).

b. What is the meaning of the numbers 9.4 and 0.043 in the equation?

c. Draw a graph of this model for the first minute after disconnecting the battery.

d. When is y less than or equal to 4.7 volts? Explain how you found this answer.

6.4 **15.** Set your calculator in scientific notation mode for this problem.

a. Use your calculator to do each division.

i. $\frac{8 \times 10^8}{2 \times 10^3}$ 4×10^5 **ii.** $\frac{9.3 \times 10^{13}}{3 \times 10^3}$ 3.1×10^{10} **iii.** $\frac{4.84 \times 10^9}{4 \times 10^4}$ 1.21×10^5 **iv.** $\frac{6.2 \times 10^4}{3.1 \times 10^8}$ 2×10^{-4}

b. Describe how you could do the calculations in 15a without using a calculator.

c. Find the answer to the quotient $\frac{4.8 \times 10^7}{8 \times 10^2}$ without using your calculator.

0.6×10^5, or 6×10^4 in scientific notation

IMPROVING YOUR REASONING SKILLS

You have learned about scientific notation in this chapter. There is another convention for writing numbers called **engineering notation.**

Engineering notation	Not in engineering notation
2.5×10^9	2500×10^3
630×10^{-3}	630×10^{-2}
12×10^0	1.5×10^5
400×10^3	0.4×10^6
10.8×10^6	1.08×10^7

1. Write a definition for engineering notation based on the numbers in the lists. If your calculator has an engineering notation mode, you can enter more numbers to help support your definition.

2. Convert these numbers to engineering notation.

a. 78,000,000 **b.** 9,450

c. 130,000,000,000 **d.** 0.0034

e. 0.31 **f.** 1.4×10^8

3. You may have seen these symbols used as shorthand for numbers:

n ("nano," or times $\frac{1}{1,000,000,000}$)
μ ("micro," or times $\frac{1}{1,000,000}$)
k ("kilo," or times 1,000)
M ("mega," or times 1,000,000)
G ("giga," or times 1,000,000,000)

Explain how engineering notation is related to these symbols.

This tool, a micrometer, is used to accurately measure very small distances. Measurements made with it may be recorded in engineering notation.

IMPROVING REASONING SKILLS

1. If students are having difficulty, suggest they get a hint from part 3. A possible definition of engineering notation that models the definition of scientific notation is $a \times 10^n$, where $1 \le a < 1000$ or $-1000 < a \le -1$ and n is a multiple of 3. As needed, point out that 0 is a multiple of 3, specifically, $3 \cdot 0$.

2. a. 78×10^6 **b.** 9.45×10^3
c. 130×10^9 **d.** 3.4×10^{-3}
e. 310×10^{-3} **f.** 140×10^6

Answers that append zeros to the first parts are acceptable. (For example, 2a may be 78.0×10^6 or 78.00×10^6.)

3. Engineering notation is related to the groups of three digits separated by commas in decimal numbers; that is, $n = 10^{-9}$, $\mu = 10^{-6}$, $k = 10^3$, $M = 10^6$, $G = 10^9$.

Students may be interested in expanding this to a project by researching careers and applications that use engineering notation.

LESSON

6.7

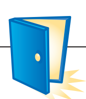
Fitting Exponential Models to Data

In broken mathematics
We estimate our prize
Vast—in its fading ratio
To our penurious eyes!
EMILY DICKINSON

Victoria Julian has been collecting data on changes in median house prices in her area over the past 10 years. She plans to buy a house 5 years from now and wants to know how much money she needs to save each month toward the down payment. How can she make an intelligent prediction of what a house might cost in the future? What assumptions will she have to make?

In the real world, situations like population growth, price inflation, and the decay of substances often tend to approximate an exponential pattern over time. With an appropriate exponential model, you can sometimes predict what might happen in the future.

In Chapter 4, you learned about fitting linear models to data. In this lesson you'll learn how to find an exponential model to fit data.

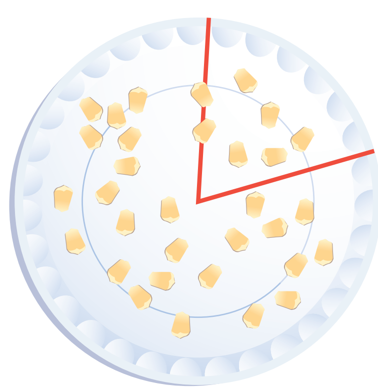


Investigation
Radioactive Decay

You will need
- a paper plate
- a protractor
- a supply of small counters

The particles that make up an atom of some elements, like uranium, are unstable. Over a period of time specific to the element, the particles will change so that the atom eventually will become a different element. This process is called **radioactive decay.**

In this investigation your counters represent atoms of a radioactive substance. Draw an angle from the center of your plate, as illustrated. Counters that fall inside the angle represent atoms that have decayed.



NCTM STANDARDS

CONTENT		PROCESS	
✓	Number	✓	Problem Solving
✓	Algebra	✓	Reasoning
✓	Geometry		Communication
✓	Measurement	✓	Connections
✓	Data/Probability	✓	Representation

LESSON OBJECTIVE

- Write exponential equations that model real-world growth and decay data

LESSON
6.7

PLANNING

LESSON OUTLINE

First day:
5 min Introduction
45 min Investigation

Second day:
20 min Sharing
10 min Example
5 min Closing
15 min Exercises

MATERIALS

- paper plates
- protractors
- small flat counters that don't roll, such as lentils, split peas, popcorn kernels, or flat candies (about 100 per group)
- 100 Grid (W or T), *optional*
- Protractors (T), *optional*
- Radioactive Decay Sample Data (W), *optional*
- Moore's Law Sample Data (W), *optional*
- Calculator Note 0H
- Fathom demonstration Fitting Exponential Models to Data, *optional*
- CBL 2 demonstration Cool It, *optional*

TEACHING

Exponential models can help make predictions in many real-world situations.

INTRODUCTION

To make an intelligent prediction, Victoria must assume that prices will continue to increase at the same rate.

Guiding the Investigation

One Step
Have students generate the data as in the investigation. Then ask them to find an equation that models the data. As you circulate, encourage groups to find ratios and come up with an exponential equation. Also ask how their angles relate to those ratios.

To avoid using materials in Steps 1–3, have each group start with a 100 grid as their starting collection of atoms and then use a calculator to generate 15 random numbers to represent decayed atoms (see Calculator Note 0H). Duplicates may occur. They cross those numbers off the grid and record how many numbers are not crossed off. They generate another 15 random numbers, cross off new ones, and again count the remaining numbers. They repeat until fewer than ten numbers remain. If you use this method, skip Steps 11 and 12.

With the plates, students should use protractors to make an angle of less than 90°, but not too small. They can approximate the center by using a ruler to find the mid-points of several diameters.

Step 2 To have the counters fall randomly, students should avoid aiming. The objective is to have the counters spread evenly and quickly.

An acceptable plan would be to count counters on the line as being within the angle and to count those that fall outside the plate, but within the extended rays of the angle, as being within the angle.

Students may get a kinesthetic feeling for decay by eating candy counters.

See page 724 for answers to Steps 1, 2, 3, 4, 9, and 10.

Step 1 Count the number of counters. Record this in a table as the number of "atoms" after 0 years of decay. Pick up all of the counters.

Step 2 Drop the counters on the plate. Count and remove the counters that fall inside the angle—these atoms have decayed. Subtract from the previous value and record the number remaining after 1 year of decay. Pick up the remaining counters.

Step 3 Repeat Step 2 until you have fewer than ten atoms that have not decayed. Each drop will represent another year of decay. Record the number of atoms remaining each time.

Procedure Note

Create a procedure for dropping counters randomly on the plate. Be sure that your method results in an approximately even distribution. Make a plan for handling counters that fall on the lines of your angle and those that miss the plate—they need to be accounted for too.

Step 4 Let x represent elapsed time in years, and let y represent the number of atoms remaining. Make a scatter plot of the data. What do you notice about the graph?

Step 5 Calculate the ratios of atoms remaining between successive years. That is, divide the number of atoms after 1 year by the number of atoms after 0 years; then divide the number of atoms after 2 years by the number of atoms after 1 year; and so on. How do the ratios compare?

Step 6 Choose one representative ratio. Explain how and why you made your choice.

Step 7 At what rate did your atoms decay?

Step 8 Write an exponential equation that models the relationship between time elapsed and the number of atoms remaining.

Step 9 Graph the equation with the scatter plot. How well does it fit the data?

Step 10 If the equation does not fit well, which values could you try to adjust to give a better fit? Record your final equation when you are satisfied.

Step 5 See table of Steps 1–3 (page 724) for sample data. The ratios should be approximately the same.

Step 6 Answers will vary. Students could give reasons for selecting the mean, the median, or another value. In these sample data, the mean is 0.802.

Step 7 For these sample data, using 0.802, the rate of decay is 19.8% per year.

Step 8 For this sample, $y = 201(1 - 0.198)^x$.

Step 11 Measure the angle on your plate. Describe a connection between your angle and the numbers in your equation.

Step 12 Based on what you've learned and the procedures outlined in this investigation, write an equation that would model the decay of 400 counters, using a central angle of 60°. What are some of the factors that might cause differences between actual data and values predicted by your equation?

Step 11 The ratio of the angle measure to 360° should be approximately the same as r. In the sample data, $\frac{68}{360} \approx 0.19$, which is close to the r-value used in Step 8.

Step 12 $y = 400\left(1 - \frac{60}{360}\right)^x$; answers will vary. Factors might include how evenly the counters are distributed on the plate, what you do when a counter is on an angle side, and how you treat counters that fall outside the plate.

Archaeologists can approximate the age of artifacts with *carbon dating.* This process uses the rate of radioactive decay of carbon-14. Carbon is found in all living things, so the amount left in a bone, for example, is an indicator of the bone's age. This is a plastic casting of a skull found in 1997 in Richland, Washington. Carbon dating has dated the skull as 9200 years old.

Step 3 Rather than counting the remaining counters each time, students may find it easier to count the number decayed and subtract from the previous amount to find the number remaining.

Step 5 Suggest that students add a column of data for recording the ratios.

Step 7 You may need to encourage students to look at their constant multiplier in the form of $(1 - r)$. That is, a ratio of $\frac{3}{4}$ would be $(1 - 0.25)$.

Steps 11 and 12 If students are not experienced at using protractors, use the Protractors transparency to model how to measure angles and how to draw an angle with a given measure.

Step 11 **[Ask]** "What is the ratio of your angle measure to the whole plate?"

Step 12 You may want to discuss theoretical value versus observed data. The starting value of the best fit might not be the actual starting value of the data.

The steps of finding an equation in the investigation provide a good method for finding an exponential equation that models data that display an exponential pattern, either increasing or decreasing. These situations are often generated recursively by multiplying by a constant ratio. Thinking of the constant multiplier in the form $1 + r$ or $1 - r$ leads to these familiar equations:

$$y = A(1 + r)^x$$
$$y = A(1 - r)^x$$

You can then fine-tune the fit of your model by slightly adjusting the values of A and r.

EXAMPLE

Every musical note has an associated frequency measured in hertz (Hz), or vibrations per second. The table shows the approximate frequencies of the notes in the octave from middle C up to the next C on a piano. (In this scale, E# is the same as F and B# is the same as C.)

Piano Notes

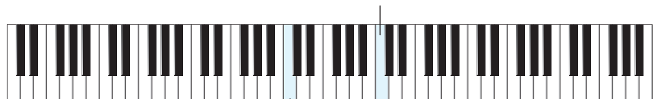
Note name	Note number above middle C	Frequency (Hz)
Middle C	0	262
C#	1	277
D	2	294
D#	3	311
E	4	330
F	5	349
F#	6	370
G	7	392
G#	8	415
A	9	440
A#	10	466
B	11	494
C above middle C	12	523

The arrangement of strings in a piano shows an exponential-like curve.

a. Find a model that fits the data.

b. Use the model to find the frequency of the note two octaves above middle C (note 24).

c. Find the note with a frequency of 600 Hz.



You might use the Fathom demonstration Fitting Exponential Models to Data to supplement the investigation. You might also have students write a lab report.

SHARING IDEAS

Have several groups present their data, scatter plots, equations, and graphs for class critique and suggestions. In the equation $y = A(1 - r)^x$, the number A is the y-intercept of the graph, so students might adjust A to shift the graph vertically. For graphs of exponential equations, vertical shifts can also be thought of as horizontal shifts. To stretch the graph, students will tinker with the decay rate, r.

Ask if anyone can explain radioactive decay and its uses. In general, a neutron of an atom is stable only when paired with a sufficient number of protons. Elements with either too few or too many protons per neutron are unstable. The nuclei of unstable elements can decay in a variety of ways. The half-life of different radioactive nuclei can be anywhere from 10^{-9} s up to 10^{20} yr.

Because carbon-14 atoms occur in all living things, knowing about their decay is extremely useful in determining the age of very old remains of life. It takes about 5730 yr for half of their excess neutrons to decay. Therefore, if one bone has half the amount of carbon-14 as a similar bone, the first bone is 5730 yr older.

[Ask] "Would there ever be zero atoms remaining?" [In the exponential equation, the value of y will never decrease all the way to zero. The counters and even protons are discrete units, however, so they will all decay eventually.]

You may want to have students generalize the steps they used in writing an equation:

1. Find the ratios between successive y-values.

2. Select a representative ratio, possibly a mean or median.

3. Think of the ratio in the form of $(1 - r)$ if the values are decreasing and $(1 + r)$ if they're increasing.

4. Use the starting value, A, to write an equation in the form $y = A(1 - r)^x$ or $y = A(1 + r)^x$.

5. Adjust the values of A and r to get a better fit.

Point out the opening quotation. It is the second stanza of an untitled poem from about 1859. The first stanza is

As by the dead we love to sit
Become so wondrous dear
As for the lost we grapple
Tho' all the rest are here.

[Language] *Penurious* means extremely stingy or poor. One interpretation is that the relationship between what we want and what we see that we currently have is an inverse relationship—the more we want, the less we see value in what we have. Another interpretation is that what we want decays exponentially with respect to how well we see what we have.

Assessing Progress
Watch for skill at handling geometric tools (ruler, protractor), collecting data, making a scatter plot, and graphing exponential equations.

▶ ***EXAMPLE***
This example shows exponential growth in a different context. You might ask students who have a musical background to elaborate on the notation. The symbol # in the table means *sharp,* or one-half step above the note; C# is one-half step above C. *Functional Melodies* (Key Curriculum Press) has additional activities involving the Pythagorean and even-tempered scales.

The solution in part c brings in a new way to solve an equation: graph the two sides of the equation and find the intersection point of the graphs.

Closing the Lesson

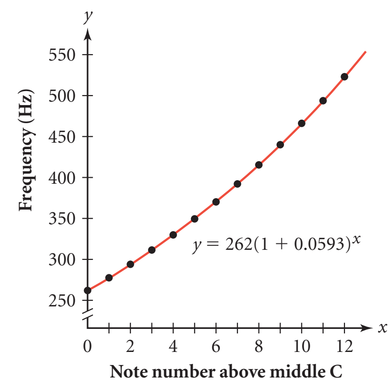
At the heart of finding the equation modeling exponential growth or decay is finding a ratio to use as a constant multiplier.

▶ ***Solution***

a. Let x represent the note number above middle C, and let y represent the frequency. A scatter plot shows the exponential-like pattern. To find the exponential model, first calculate the ratios between successive data points. The mean of the ratios is 1.0593. So the frequency of the notes increases by about 5.93% each time you move up one note on the keyboard. The starting frequency is 262 Hz. So an equation is

$$y = 262(1 + 0.0593)^x$$

The graph shows a very good fit.



b. To find the frequency of the C two octaves above middle C (note 24), substitute 24 for x in the model.

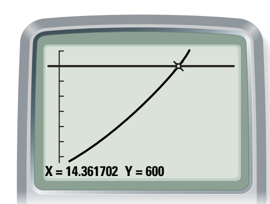
$$y = 262(1 + 0.0593)^{24} \approx 1044$$

By this model, the frequency of note 24 is 1044 Hz.

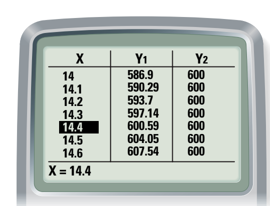
c. To find the note with a frequency of 600 Hz, substitute 600 for y in the model.

$$600 = 262(1 + 0.0593)^x$$

Enter $262(1 + 0.0593)^x$ into Y_1 and 600 into Y_2 on your calculator. Graph both equations and trace to approximate the intersection point. Or you could look at a table to see where $Y_1 = Y_2$.



[0, 18, 10, 250, 650, 50]



Both the graph and the table show an x-value between 14 and 15. The 14th note above middle C is a D and the 15th note is a D#. Because the piano notes correspond only to whole numbers, you cannot make a note with a frequency of 600 Hz on this piano.

If you wanted to find the frequency of notes below middle C, you would need to use negative values for x. The frequencies found using this equation will be fairly accurate because the data fit the equation so well. If the piano were very out of tune, the equation probably would not fit so nicely, and the model might be less valuable for predicting.

Music CONNECTION

Before the 17th century, there were many ways to tune an instrument. The most popular, developed by the ancient Greek philosopher Pythagoras, used different tuning ratios between each pair of adjacent notes. This made some scales, like the scale of C, sound good but others, like the scale of A-flat, sound bad. Modern Western tuning now uses *even temperament,* based on an equal tuning ratio between adjacent notes, which leads to an exponential model.

Exercises

You will need your graphing calculator for Exercises **5, 6, 8, 9, 10,** and **11.**

Practice Your Skills

1. Rewrite each value as either $1 + r$ or $1 - r$. Then state the rate of increase or decrease as a percent.

a. 1.15 @ $1 + 0.15$; rate of increase: 15% **b.** 1.08 $1 + 0.08$; rate of increase: 8% **c.** 0.76 @ $1 - 0.24$; rate of decrease: 24%

d. 0.998 $1 - 0.002$; rate of decrease: 0.2% **e.** 2.5 $1 + 1.5$; rate of increase: 150%

2. Use the equation $y = 47(1 - 0.12)^x$ to answer each question.

a. Does this equation model an increasing or decreasing pattern? ⓗ decreasing

b. What is the rate of increase or decrease? 12%

c. What is the y-value when x is 13? $y \approx 8.92$

d. What happens to the y-values as the x-values get very large? The y-values approach zero.

3. Write an equation to model the growth of an initial deposit of \$250 in a savings account that pays 4.25% annual interest. Let B represent the balance in the account, and let t represent the number of years the money has been in the account. @ $B = 250(1 + 0.0425)^t$

4. Use the properties of exponents to rewrite each expression with only positive exponents.

a. $4x^3 \cdot (3x^5)^3$ $108x^{18}$ **b.** $\dfrac{60x^8y^4}{15x^3y}$ @ $4x^5y^3$ **c.** $3^2 \cdot 2^3$ 72 **d.** $\dfrac{(8x^3)^2}{(4x^2)^3}$ @ 1

e. $x^{-3}y^4$ $\dfrac{y^4}{x^3}$ **f.** $(2x)^{-3}$ $\dfrac{1}{8x^3}$ **g.** $2x^{-3}$ $\dfrac{2}{x^3}$ **h.** $\dfrac{2x^{-4}}{(3y^2)^{-3}}$ $\dfrac{54y^6}{x^4}$

Reason and Apply

5. Mya placed a cup of hot water in a freezer. Then she recorded the temperature of the water each minute.

Water Temperature

Time (min) x	0	1	2	3	4	5	6	7	8	9	10
Temperature (°C) y	47	45	43	41.5	40	38.5	37	35.5	34	33	31.5

[Data sets: **FZTIM, FZTMP**]

a. Find the ratios between successive temperatures. @

b. Find the mean of the ratios in 5a. @ approximately 0.96

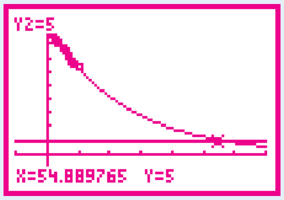
c. Write the ratio from 5b in the form $1 - r$. @ $1 - 0.04$

d. Use your answer from 5c and the starting temperature to write an equation in the form $y = A(1 - r)^x$. @ $y = 47(1 - 0.04)^x$

e. Graph your equation with a scatter plot of the data. Adjust the values of A or r until you get a satisfactory fit.

f. Use your equation to predict how long it will take for the water temperature to drop below 5°C.

5f. 55 min



$[-10, 70, 10, -10, 50, 5]$

Building Understanding

The exercises provide practice in developing exponential models.

Assigning Homework

Essential	1–3, 5–7
Performance assessment	8, 12
Portfolio	5, 6
Journal	11
Group	5, 6, 9
Review	4, 12–14

Helping with the Exercises

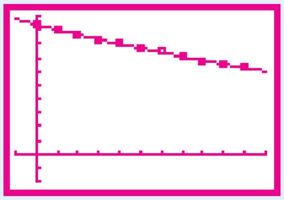
Exercise 1e Some students may not realize that a number that's more than 1 is more than 100%.

Exercise 2d This question brings in the idea of limit, as discussed in Chapter 0. **[Ask]** "Why do powers of positive numbers less than 1 get smaller and smaller?" [Multiplying a positive number by a positive number less than 1 yields a smaller positive number.]

Exercises 5 and 6 Students can use a CBL 2 to collect data for these exercises. You can use the CBL 2 demonstration Cool It as an extension for Exercise 5.

5a. The ratios are 0.957, 0.956, 0.965, 0.964, 0.963, 0.961, 0.959, 0.958, 0.971, and 0.955.

5e.



$[-1, 11, 1, -10, 50, 5]$

Adjustments to A or r are not necessary—the fit is good as is.

Exercise 5f Ask if the temperature will ever drop to freezing (0°C). Theoretically, an exponential expression will never actually get to zero, though in real life water freezes.

6a. possible answer: $y = 431(1 - 0.26)^x$, where 0.26 is derived from the mean ratio of about 0.74

6b. With each layer of plastic, the amount of light is reduced 26%.

6c. With 9 layers, the reading would be below 30.

Exercise 7 This data set is artificial. You might encourage students to research similar data for their own town or make their own projections.

Exercise 8 "Magic pot" stories have developed in the folklore of several cultures, including India, Italy, and China. Titles include *The Magic Porridge Pot, The Magic Pasta Pot,* and simply *The Magic Pot. The Sorcerer's Apprentice* has a similar theme.

8b. $y = 2(1 + 0.5)^{0.5} \approx 2.45$; approximately 2.45 L

8c. approximately 115 L

6. In science class Phylis used a light sensor to measure the intensity of light (in lumens per square meter, or lux) that passes through layers of colored plastic. The table below shows her readings.

Light Experiment

Number of layers	0	1	2	3	4	5	6
Intensity of light (lux)	431	316	233	174	128	98	73

[Data sets: LTLAY, LTINT]

a. Write an exponential equation to model Phylis's data. Let x represent the number of layers, and let y represent the intensity of light in lux. ⓗ

b. What does your r-value represent?

c. If Phylis's sensor cannot register readings below 30 lux, how many layers can she add before the sensor stops registering?

7. Suppose that on Sunday you see 32 mosquitoes in your room. On Monday you count 48 mosquitoes. On Tuesday there are 72 mosquitoes. Assume that the population will continue to grow exponentially.

a. What is the percent rate of growth? ⓐ 50%

b. Write an equation that models the number of mosquitoes, y, after x days. $y = 32(1 + 0.5)^x$

c. Graph your equation and use it to find the number of mosquitoes after 5 days, after 2 weeks, and after 4 weeks. 243 mosquitoes; 9,342 mosquitoes; 2,727,126 mosquitoes

d. Name at least one real-life factor that would cause the population of mosquitoes not to grow exponentially. Answers will vary. Possibilities include lack of resources, overcrowding, and extermination.

8. There are many stories in children's literature that involve magic pots. An Italian variation goes something like this: A woman puts a pot of water on the stove to boil. She says some special words, and the pot begins filling with pasta. Then she says another set of special words, and the pot stops filling up.

Suppose someone overhears the first words, takes the pot, and starts it in its pasta-creating mode. Two liters of pasta are created. Then the pot continues to create more pasta because the impostor doesn't know the second set of words. The volume continues to increase 50% per minute.

a. Write an equation that models the amount of pasta in liters, y, after x minutes. ⓐ $y = 2(1 + 0.5)^x$

b. How much pasta will there be after 30 seconds?

c. How much pasta will there be after 10 minutes?

d. How long, to the nearest second, will it be until the entire house, which can hold 450,000 liters, is full of pasta? after about 30.4 min, or 30 min 24 s

9. **APPLICATION** Recall Victoria from the opening of this lesson. She has collected this table of data on median house prices for her area.

Median House Prices

Year	Years since 2000	Median price ($)
2000	0	135,500
2001	1	144,000
2002	2	152,500
2003	3	161,500
2004	4	171,500
2005	5	181,500
2006	6	192,250

[Data sets: **HSEYR, HSEYS, HSEPR**]

a. Define variables and find an exponential equation to model Victoria's data. @

b. Victoria plans to buy a house 5 years from now. What median price should she expect then?

c. Victoria plans to make a down payment of 10% of the purchase price. Based on your answer to 9b, how much money will she need for her down payment?

d. If Victoria saves the same amount each year for the next 5 years (without interest), how much will she need to save each month for her down payment?

9a. Possible answer: Let x represent years since 2000 and y represent median price in dollars. An equation is $y = 135{,}500(1 + 0.06)^x$, where 0.06 is derived from the mean ratio of about 1.06.

10. The equation $y = 262(1 + 0.0593)^x$ models the frequency in hertz of various notes on the piano, with middle C considered as note 0. The average human ear can detect frequencies between 20 and 20,000 hertz. If a piano keyboard were extended, the highest and lowest notes audible to the average human ear would be how far above and below middle C? @ Note 75 above middle C (a D#) would be the highest audible note; note −44 (an E 44 notes below middle C) would be the lowest audible note.

Exercise 10 [Language] *Audible* means capable of being heard. A standard keyboard has 88 keys.

11. ***Mini-Investigation*** In this exercise you will explore the equation $y = 10(1 - 0.25)^x$.

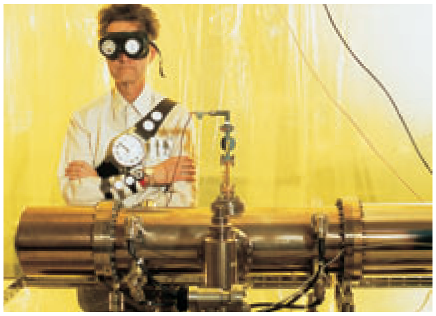
a. Find y for some large positive values of x, such as 100, 500, and 1000. What happens to y as x gets larger and larger? y gets closer and closer to zero.

b. The calculator will say y is 0 when x equals 10,000. Is this correct? Explain why or why not. No, because y can never equal zero. The number is just smaller than the calculator is able to represent.

c. Find y for some large negative values of x, such as −100, −500, and −1000. What happens to y as x moves farther and farther from 0 in the negative direction? y approaches infinity.

Review

6.6 12. Very small amounts of time much less than a second have special names. Some of these names may be familiar to you, such as a millisecond, or 0.001 second. Have you heard of a nanosecond or a microsecond? A nanosecond is 1×10^{-9} second, and a microsecond is 1×10^{-6} second. How many nanoseconds are in a microsecond? 1000 nanoseconds per microsecond



This is Jim Gray, keeper of the NBS-4 atomic clock. Atomic clocks gain or lose less than a microsecond each year. For more information, see the links at www.keymath.com/DA .

Exercise 12 Students may recall *nano* and *micro* from Lesson 6.6, Improving Your Reasoning Skills, about engineering notation. As needed, offer help with subtracting a negative number.

6.2 13. **APPLICATION** Lila researched tuition costs at several colleges she's interested in. The data are listed below. Costs are predicted to go up 3.7% each year.

[Data set: **TUITN**]

$2,860 $3,580 $8,240 $9,460
$11,420 $22,500 $26,780

a. What will the costs be next year? Answers are rounded to the nearest $10: $2,970, $3,710, $8,540, $9,810, $11,840, $23,330, $27,770.

b. Find the estimated cost for each school five years from now.
Answers are rounded to the nearest $10: $3,430, $4,290, $9,880, $11,340, $13,690, $26,980, $32,110.

9b–d. Answers will vary depending on year. See the table for possible answers.

Year	9b. Median price ($)	9c. Down payment ($)	9d. Savings ($/mo)
2011	257,219	25,722	429
2012	272,653	27,265	454
2013	289,012	28,901	482
2014	306,352	30,635	511
2015	324,734	32,473	541
2016	344,218	34,422	574
2017	364,871	36,487	608

Exercise 14 A *joule* is the SI (International System of Units) unit for the amount of work done by a force of 1 newton acting over a distance of 1 m. It is equivalent to 1 watt-second, so a 100-watt bulb burning for an hour produces 360,000 joules of energy.

Other Laws
Machrone's Law states that the machine you want always costs \$5,000 (for example, the purchase price of transistors is cut in half every 2 yr, so as the number of transistors doubles, the price stays the same). Rock's Law states that the cost of equipment to build semiconductors doubles every 4 yr.

6.4 **14.** One of the most famous formulas in science is

$$E = mc^2$$

This equation, formulated by Albert Einstein in 1905, describes the relationship between mass (m, measured in kilograms) and energy (E, measured in joules) and shows how they can be converted from one to the other. The variable c is the speed of light, 3×10^8 meters per second. How much energy could be created from a 5-kilogram bowling ball? Express your answer in scientific notation.

$5(3 \times 10^8)^2 = 5(9 \times 10^{16}) = 45 \times 10^{16}$, or 4.5×10^{17} joules

James Joule (1818–1889) was one of the first scientists to study how energy was related to heat. At the time of his experiments, many scientists thought heat was a gas that seeped in and out of objects. The SI (metric) unit of energy was named in his honor.

project

MOORE'S LAW

In 1965 Gordon Moore, the co-founder of Intel Corporation, observed that the number of transistors on a computer chip doubled approximately every 2 years. Because a computer processor's speed and power are proportional to the number of transistors on it, computers should get twice as powerful every 2 years.

Has "Moore's Law" come true since 1965? Research technical specifications for various computer processors and find an exponential model that relates time and number of transistors. You can research data in magazines or at www.keymath.com/DA. How many years or months has it taken for computers to double in power? At what rate has the power of computers increased each year?

Your project should include

- A scatter plot of your data.
- An exponential equation that models the data and an explanation of each number and variable in your equation.
- A report summarizing your findings.

You may want to research news items that give recent projections and see if computer chip manufacturers are continuing to meet or exceed Moore's Law. You may also want to research other theories on computer production that examine variables such as purchase price or equipment required for production.

Fathom
With Fathom you can easily graph an exponential equation through data points. You can use a slider to make small adjustments in your equation until it fits. You can graph multiple models each with its own slider to compare different exponential equations.

Supporting the project

MOTIVATION

This project challenges students by requiring evaluation of data whose x-values are not likely to be sequential. (Students may use the Moore's Law Sample Data worksheet.)

OUTCOMES

- The report gives evidence that the relationship between time and number of transistors is roughly exponential, as Moore's Law predicts.
- Analysis (for the sample data) shows that a good model is $y = 2300(1 + 0.38)^x$, where x is time in years since 1971 and y is number of transistors.
- The report shows that, by the same model, the number of transistors doubles in a little over 2 yr.
- The presentation may show an alternative model: $y = 2300(2)^{x/2.15}$. Here, the base, 2, represents doubling, and dividing x by 2.15 means that the number of transistors will double in 2.15 yr.

LESSON

6.8

Activity Day

Decreasing Exponential Models and Half-Life

In Lesson 6.7, you learned that data can sometimes be modeled using the exponential equation $y = A(1 - r)^x$. In this lesson you will do an experiment, write an equation that models the decreasing exponential pattern, and find the **half-life**—the amount of time needed for a substance or an activity to decrease to one-half its starting value. To find the half-life, approximate the value of x that makes y equal to $\frac{1}{2} \cdot A$.

keymath.com/DA

In the previous investigation, if your plate was marked with a 72° angle and you started with 200 "atoms," a model for the data could be $y = 200(1 - 0.20)^x$. This is because the ratio of the angle to the whole plate is $\frac{72}{360}$, or 0.20. To determine the half-life of your atoms, you would need to find out how many drops you would expect to do before you had 100 atoms remaining. Hence, you could solve the equation $100 = 200(1 - 0.20)^x$ for x using a graph or a calculator table. The x-value in this situation is approximately 3, which means your atoms have a half-life of about 3 years.

Technology CONNECTION

You can see simulations of atomic half-life with a link at www.keymath.com/DA.

Activity

Bouncing and Swinging

You will need

- a motion sensor
- a meterstick
- a ball
- string
- a soda can half-filled with water

There are two experiments described in this activity. Each group should choose at least one, collect and analyze data, and prepare a presentation of results.

LESSON

6.8

PLANNING

LESSON OUTLINE

One day:

5 min	Introduction
35 min	Activity
10 min	Sharing

MATERIALS

- balls
- metersticks
- motion sensor, *optional*
- soda cans half-filled with water
- string
- Bounce Sample Data (W), *optional*
- Pendulum Sample Data (W), *optional*
- Calculator Note 6D

TEACHING

Exponential decay equations can model processes that slow down.

Guiding the Activity

It is not mandatory to have all of the materials. The motion sensor can be supplanted with careful low-tech data collection. The bouncing ball is needed only for Experiment 1, and the string and can are needed only for Experiment 2. In place of tying a string around the pull tab of a soda can, you might tie the string around the neck of a water bottle.

If the materials in general are problematic, try the 100-grid alternative explained in Lesson 6.7. Sample data for each experiment are available on the worksheets and on the Programs and Data CD.

NCTM STANDARDS

CONTENT		PROCESS	
✓	Number		Problem Solving
✓	Algebra	✓	Reasoning
	Geometry		Communication
✓	Measurement	✓	Connections
✓	Data/Probability	✓	Representation

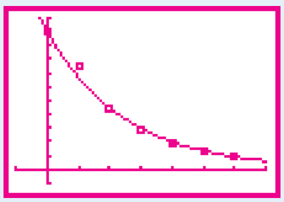
LESSON OBJECTIVE

- Write exponential equations that model real-world decay data

Step 3 The location of the pendulum bob is harmonic, but its maximum distance from the resting position is roughly exponential.

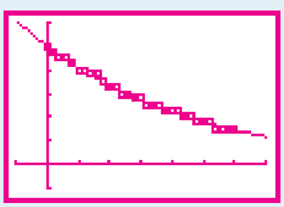
Step 4 Students who collect pendulum data by eye will need to account for collecting data every fifth swing. One option is $y = A(1 + r)^{x/5}$, where x is the number of swings.

Step 4 for sample data:
Exp. 1: $y = 1(1 - 0.33)^x$



$[-1, 7, 1, -0.1, 1.1, 0.1]$

Exp. 2: $y = 0.50(1 - 0.04)^x$



$[-5, 35, 5, -0.1, 0.6, 0.1]$

Step 6 You might want to introduce the equation $y = A\left(\frac{1}{2}\right)^{x/t}$, where t is the half-life. Students can see that the graph of this equation is similar to that of their equation in the form $y = A(1 - r)^x$. Ask them to think about why the graphs are the same. [Looking at special cases, when $x = t$, the quantity A is multiplied by $\frac{1}{2}$. When $x = 2t$, $\left(\frac{1}{2}\right)A$ is multiplied by $\frac{1}{2}$, to make $\frac{1}{4}A$. And so on. Symbolically, by definition of half-life, $b^t = \frac{1}{2}$, so raising both sides to the $\frac{x}{t}$ power gives $b^x = \left(\frac{1}{2}\right)^{x/t}$.]

SHARING IDEAS

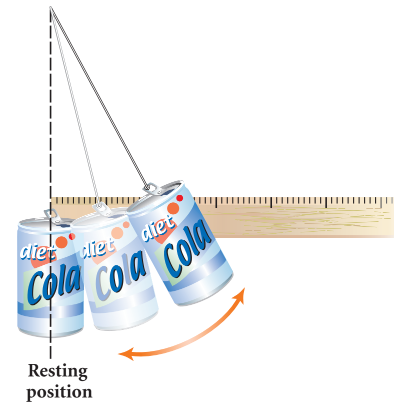
Groups can share their summaries from Step 7. The class can discuss the reasons for differences in data and equations that model the data.

Step 1 Select one of these two experiments.

Experiment 1: Ball Bounce

Drop a ball from a height of about 1 m and measure its rebound height for at least 6 bounces. You can collect data "by eye" using a meterstick, or you can use a motion sensor. [▶ See Calculator Note 6D.◀] If you use a motion sensor, hold it $\frac{1}{2}$ m above the ball and collect data for about 8 s; trace the resulting scatter plot of data points to find the maximum rebound heights.

Experiment 2: Pendulum Swing



Make a pendulum with a soda can half-filled with water tied to at least 1 m of string—use the pull tab on the can to connect it to the string. Pull the can back about $\frac{1}{2}$ m from its resting position and then release it. Measure how far the can swings from the resting position for several swings. You can collect data "by eye" using a meterstick (you may have to collect data for every fifth swing in this case), or you can use a motion sensor. [▶ See Calculator Note 6D.◀] If you use a motion sensor, position it 1 m from the can along the path of the swing; the program will collect the maximum distance from the resting position for 30 swings.

Step 2 Set up your experiment and collect data. Based on your results, you might want to modify your setup and repeat your data collection.

Step 3 The scatter plot should show an exponential pattern.

Step 3 Define variables and make a scatter plot of your data on your calculator. (If you used a motion sensor, you should have this already.) Draw the scatter plot accurately on your paper. Does the graph show an exponential pattern?

Step 4 Find an equation in the form $y = A(1 - r)^x$ that models your data. Graph this equation with your scatter plot and adjust the values if a better fit is needed.

Step 5 One method is to graph $Y_2 = \frac{1}{2} \cdot A$ and find the intersection with $Y_1 = A(1 - r)^x$. Using sample data, half-lives are Exp. 1: about 2 bounces, Exp. 2: about 17 swings.

Step 5 Find the half-life of your data. Explain what the half-life means for the situation in your experiment. (Read page 381 to review the calculation of half-life.)

Step 6 Find the y-value after 1 half-life, 2 half-lives, and 3 half-lives. How do these values compare? With each consecutive half-life, the value of y will be half the previous value of y.

Step 7 Write a summary of your results. Include descriptions of how you found your exponential model, what the rate r means in your equation, and how you found the half-life. You might want to include ways you could improve your setup and data collection.

In the real world, eventually your ball will stop bouncing or your pendulum will stop swinging. Your exponential model, however, will never reach a y-value of zero. Remember that any mathematical model is, at best, an approximation and will therefore have limitations.

Assessing Progress

Watch for students' ability to collect data systematically, to define variables, to make a scatter plot, to find common ratios and write an appropriate exponential decay equation, and to find the half-life.

Closing the Lesson

As needed, point out that the term *exponential decay* refers to slowing processes other than radioactive decay.

CHAPTER 6 REVIEW

You started this chapter by creating sequences that increase or decrease when you multiply each term by a constant factor. Repeated multiplication causes the rate of change between successive terms to increase or decrease. So the graphs of these sequences curve, getting steeper and steeper or less and less steep. You then discovered that **exponential equations** model these sequences, in which the constant multiplier is the **base** and the number of the term in the sequence is the **exponent.**

By writing exponential expressions in both **expanded form** and **exponential form,** you learned the **multiplication, division,** and **power properties of exponents,** and you explored the meanings of zero and negative exponents. You applied these properties to **scientific notation,** a way to express numbers with powers of 10.

When modeling data, you can often use an equation to make predictions. You now have two kinds of models for real-world data—linear equations and exponential equations. Many real-world quantities that increase can be modeled as **exponential growth** with an equation in the form $y = A(1 + r)^x$. You can model many quantities that decrease, like **radioactive decay,** with an equation in the form $y = A(1 - r)^x$.

EXERCISES

You will need your graphing calculator for Exercises **2, 3,** and **10.**

@ Answers are provided for all exercises in this set.

1. Write each number in exponential form with base 3.

a. 81 3^4 **b.** 27 3^3 **c.** 9 3^2

d. $\frac{1}{3}$ 3^{-1} **e.** $\frac{1}{9}$ 3^{-2} **f.** 1 3^0

2. Use the properties of exponents to rewrite each expression. Your final answer should have only positive exponents. Use calculator tables to check that your expression is equivalent to the original expression.

a. $\frac{x \cdot x \cdot x}{x}$ x^2 **b.** $2x^{-1}$ $\frac{2}{x}$ **c.** $\frac{6.273x^8}{5.1x^3}$ $1.23x^5$ **d.** 3^{-x} $\frac{1}{3^x}$

e. $3x^0$ 3 **f.** $x^2 \cdot x^5$ x^7 **g.** $\left(3^4\right)^x$ 3^{4x} **h.** $\frac{1}{x^{-2}}$ x^2

3. Consider this exponential equation:

$y = 300(1 - 0.15)^x$

a. Invent a real-world situation that you can model with this equation. Give the meaning of 300 and of 0.15 in your situation.

b. What would the inequality $75 \leq 300(1 - 0.15)^x$ mean for your situation in 3a? Answers will vary given the context of 3a.

c. Find all integer values of x such that $75 \leq 300(1 - 0.15)^x$. $x \leq 8$ or $0 \leq x \leq 8$ (some integers may be excluded by the real-life situation).

Helping with the Exercises

Exercise 3 This exercise reviews inequality along with exponential equations.

3a. Possible answer: A $300 microwave depreciates at a rate of 15% per year.

3b. the years (x) for which the depreciating value of the microwave is at least $75

CHAPTER REVIEW 6

PLANNING

LESSON OUTLINE

One day:

10 min	Introduction
15 min	Exercises
10 min	Checking work
15 min	Student self-assessment

REVIEWING

Refer students to Lesson 6.2, Exercise 13. **[Ask]** "What is the equation asked for in 13a?" [$y = 5000(1 + 0.05)^x$] "By this model, how much will the car be worth in 5 years?" [$6,381] "How much will it be worth 3 years after that?" [$7,387] "What about 2 years before that?" [After 6 years from the time she purchased the car, it will be worth $6,700.] Review the addition of exponents when multiplying powers of the same base and the subtraction of exponents when dividing powers of the same base. Then ask how much the car was worth 5 years ago. Model the problem using negative exponents, and discuss the fact that a model is not always accurate. **[Ask]** "What would the car be worth 5 years from now if, contrary to Shawna's hopes, it depreciates at 7% per year instead of appreciating?" [$3,478]

ASSIGNING HOMEWORK

You might assign the even-numbered problems (2–10) for homework and allow students to work on the odd-numbered problems in class while you take time to work individually with students who have questions.

4. Answers will vary. Possible answer: $\frac{3^x}{3^x} = 3^{x-x} = 3^0$. The result of any number divided by itself is 1.

4. Proaga says, "Three to the power of zero must be zero. An exponent tells you how many times to multiply the base, and if you multiply zero times you would have nothing!" Give her a convincing argument that 3^0 equals 1.

5. For each table, find the value of the constants A and r such that $y = A(1 + r)^x$ or $y = A(1 - r)^x$. Then use your equations to find the missing values.

a. $y = 200(1 + 0.4)^x$

x	y
0	200
1	280
2	392
3	548.8
4	768.32
5	1075.648
6	1505.9072

b. $y = 850(1 - 0.15)^x$

x	y
−2	1176.4706
−1	1000.0000
0	850
1	722.5
2	614.125
3	522.00625
4	443.7053

6. Convert each number from scientific notation to standard notation, or vice versa.

a. -2.4×10^6 −2,400,000

b. 3.25×10^{-4} 0.000325

c. 37,140,000,000 3.714×10^{10}

d. 0.00000008011 8.011×10^{-8}

Exercise 7 Encourage students to use dimensional analysis.

7. A person blinks about 9365 times a day. Each blink lasts about 0.15 second. If one person lives 72 years, how many years will be spent with his or her eyes closed while blinking? Write your answer in scientific notation. approximately 1.17×10^0 yr

One of the purposes of blinking is to spread tears over the eye. The American photographer Man Ray (1890–1976) is well-known for this photo titled *Glass Tear.*

Exercise 8 Encourage a variety of approaches, including recursion. This exercise provides a good chance to ask students about reasonable values of y. The equation $y = 1.00(1 + 0.03)^x$ can yield any positive real value of y. But if, for example, the smallest denomination of coin accepted by the machine is a nickel, then only multiples of 0.05 fit the real-world situation.

8. APPLICATION In 2004, a can of soda cost \$1.00 in a vending machine. If prices increase about 3% per year, in what year will the cost first exceed \$2? after 24 yr, or in 2028

9. Classify each equation as true or false. If false, explain why and change the right side of the equation to make it true.

a. $(3x^2)^3 = 9x^6$

b. $3^2 \cdot 2^3 = 6^5$

c. $2x^{-2} = \frac{1}{2x^2}$

d. $\left(\frac{x^2}{y^3}\right)^3 = \frac{x^5}{y^6}$

9a. False; 3 to the power of 3 is not 9; $27x^6$.

9b. False; you can't use the multiplication property of exponents if the bases are different; $9 \cdot 8$, or 72.

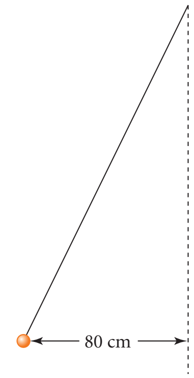
9c. False; the exponent −2 applies only to x; $\frac{2}{x^2}$.

9d. False; the power property of exponents says to multiply exponents; $\frac{x^6}{y^9}$.

10. APPLICATION A pendulum is pulled back 80 centimeters horizontally from its resting position and then released. The maximum distance of the swing from the resting position is recorded after each minute for 5 minutes.

Pendulum Swings

Time elapsed (min)	0	1	2	3	4	5
Maximum distance from resting position (cm)	80	66	55	46	38	32



a. Define variables and write an equation that models the maximum distance of the swing after each minute.

b. What is the maximum distance from the resting position after 9 minutes? approximately 15.0 cm

c. After how many minutes will the maximum distance from the resting position be less than 5 centimeters? 15 min

Exercise 10 As mentioned in Lesson 6.8, pendulum motion is fundamentally harmonic. But the aspect that this exercise simulates can be roughly modeled with an exponential equation. However, that doesn't mean it is truly exponential.

10a. Possible answer: $y = 80(1 - 0.17)^x$, where x is the time elapsed in minutes and y is the maximum distance in centimeters; $(1 - 0.17)$ is derived from the mean ratio of approximately 0.83.

TAKE ANOTHER LOOK

Scientific notation gives scientists and mathematicians one way to express extremely large and extremely small numbers. Sometimes scientists focus on only the power of 10 to describe size or quantity, calling this the **order of magnitude.**

Consider that the average distance from Earth to the Sun is 9.29×10^7 miles. Unless a scientist is going to calculate with this figure, she may simply say the distance in miles from Earth to the Sun is *on the order of* 10^7. By stating only the power of 10, what range of values is the scientist including?

Order of magnitude is also used to compare numbers. Suppose a sample of bacteria grows from several hundred to several thousand cells overnight. How many times larger is the sample now? A scientist may say the number of cells in the sample *increased by one order of magnitude,* because $\frac{10^3}{10^2}$ equals 10^1. What would the scientist say when the sample grows from several hundred cells to several hundred thousand cells? What fraction of cells would remain if the sample *decreased* by two orders of magnitude? (*Note:* The units must be equal to compare orders of magnitude.)

Think about the relative size of our universe as you answer these questions:

1. Explain what it means for the typical size of a cell in meters to be on the order of 10^{-6}.
2. Explain what it means for the length of a cow in meters to be on the order of 10^0.
3. The distance in meters from Earth to the nearest star (other than the Sun) is on the order of 10^{17}. Is it correct to compare the distance from Earth to the Sun and the distance from Earth to the nearest star as an increase by 10 orders of magnitude, because $\frac{10^{17}}{10^7}$ equals 10^{10}?

Take Another Look

Students may better understand order of magnitude if they relate it to maximum place value and can describe orders of magnitude with words. (For example, 10^7 means the maximum place value is the ten millions.)

By stating the order of 10^7, the scientist is including a range of values of at least 10,000,000 and less than 100,000,000.

If a sample grows from several hundred cells to several thousand cells, it has increased roughly 10 times.

If the sample grows from several hundred cells to several hundred thousand cells, it has increased by three orders of magnitude. If the cells decrease by two orders of magnitude, about $\frac{1}{100}$ remain.

1. The size of a cell includes a range of values of at least 0.000001 and less than 0.00001.

2. The length of a cow is at least 1 m and less than 10 m.

3. This is incorrect because the units are not equivalent (meters versus miles).

4. This is an increase by 26 orders of magnitude.

An increase by 100% does not represent an increase in order of magnitude. An increase of 100% means the quantity doubles, whereas an increase in an order of magnitude means that the quantity is multiplied by about 10.

ASSESSING

Use assessment of students' group investigation skills, projects, and self-assessment, along with a written test, to measure students' progress. For written assessment, use two or three Constructive Assessment items, or Form A or B of the Chapter Test from Assessment Resources, or use the test generator to construct a written test based on the lessons your class completed.

FACILITATING SELF-ASSESSMENT

To help students complete the portfolio described in Assessing What You've Learned, suggest that they consider for evaluation their work on Lesson 6.1, Exercise 9; Lesson 6.2, Exercise 9 or 12; Lesson 6.3, Exercise 11 or 13; Lesson 6.4, Exercise 7 or 14; Lesson 6.5, Exercise 8 or 10; Lesson 6.6, Exercise 12; and Lesson 6.7, Exercise 5 or 6.

4. The diameter in meters of the Milky Way galaxy is 10^{20}. Describe the increase in order of magnitude between the size of a cell and the size of the galaxy.

When something increases 100%, should it be described as an increase in order of magnitude? Give an example to support your conclusion.

Assessing What You've Learned

WRITE IN YOUR JOURNAL Add to your journal by considering one of these prompts:

- Why is scientific notation convenient for writing extremely large or extremely small numbers? Are there numbers that you find to be less convenient to write in scientific notation? Does scientific notation help you to understand why our standard number system is called a "base 10" system?
- Compare and contrast linear and exponential data. How do the graphs differ? If you weren't specifically told to find either a linear or an exponential equation to fit a graph of data, how would you decide which to try? How do the methods of fitting linear and exponential models compare?

PERFORMANCE ASSESSMENT Show a classmate, a family member, or your teacher that you know how to find an exponential model in the form $y = A(1 + r)^x$. You may want to go back and use the data sets from Lesson 6.7 or Lesson 6.8, or use data that you have collected from a project. Explain why you think the data are exponential, and when and why you would want to adjust the value of A or r.

GIVE A PRESENTATION Review the properties of exponents that you learned in this chapter. Think about the techniques you have used to remember these properties, or ask your peers, teachers, or family members how they remember these properties. Prepare a presentation for your class and demonstrate the memory methods you have learned. Your presentation will help your classmates remember the properties of exponents too!